STAT253/317 Lecture 10

Cong Ma

Chapter 6 Continuous-Time Markov Chains

6.2 Continuous-Time Markov Chains (CTMC)

Definitions. A stochastic process $\{X(t), t \geq 0\}$ with state space $\mathcal{X}$ is called a *continuous-time Markov chain* if for any two states i, $j \in \mathcal{X}$,

$$\begin{aligned}
&\mathrm{P}(\underbrace{X(t+s)=j}_{\text{future}} \,|\, \underbrace{X(s)=i}_{\text{present}}, \underbrace{X(u)=x(u), \text{for } 0 \leq u < s}_{\text{past}}) \\
&= \mathrm{P}(\underbrace{X(t+s)=j}_{\text{future}} \,|\, \underbrace{X(s)=i}_{\text{present}})
\end{aligned}$$

If $\mathrm{P}(X(t+s)=j|X(s)=i)$ does not depend on s for all $i, j \in \mathcal{X}$, then it is denoted as

$$P_{ij}(t) = \mathrm{P}(X(t+s)=j|X(s)=i),$$

and we say the CTMC is *homogeneous* in time.

In STAT253/317, we focus on homogeneous CTMC only.

Exponential Waiting Time

Let $\{X(t), t \geq 0\}$ be a homogeneous continuous-time Markov chain. For $i \in \mathcal{X}$, let T_i denote the amount of time that $X(t)$ stays in state i before making a transition into a different state.

Claim: T_i has the *memoryless property*.

$$
\begin{aligned}
&\mathrm{P}(T_i \geq t + s | T_i \geq s) \\
&= \mathrm{P}(X(u) = i, \text{ for } s \leq u \leq s+t | X(u) = i, \text{ for } 0 \leq u \leq s) \\
&= \mathrm{P}(X(u) = i, \text{ for } s \leq u \leq s+t | X(s) = i) \quad \text{(Markov property)} \\
&= \mathrm{P}(X(u) = i, \text{ for } 0 \leq u \leq t | X(0) = i) \qquad \text{(time homogeneity)} \\
&= \mathrm{P}(T_i \geq t) \quad \Rightarrow \quad \text{So } T_i \text{ is memoryless.}
\end{aligned}
$$

Recall that the exponential distribution is the only continuous distribution having the memoryless property.
Thus $T_i \sim Exp(\nu_i)$ for some rate ν_i.

An Alternative Definition of CTMC

A stochastic process $\{X(t), t \geq 0\}$ with state space $\mathcal{X}$ is a *continuous-time Markov chain* if

- (exponential waiting time) when the chain reaches a state i, the time it stays at state $i \sim Exp(\nu_i)$, where ν_i is the transition rate at state i
- (embedded with a discrete time Markov chain) when the process leaves state i, it enters anther state j with probability P_{ij}, such that

$$P_{ii} = 0, \quad \sum\nolimits_{j \in \mathcal{X}} P_{ij} = 1 \quad \text{for all } i, j \in \mathcal{X}.$$

Remark: The amount of time T_i the process spends in state i, and the next state visited, must be independent. For if the next state visited were dependent on T_i, then information as to how long the process has already been in state i would be relevant to the prediction of the next state—and this contradicts the Markovian assumption.

6.3 Birth and Death Processes

Let $X(t)$ = the number of people in the system at time t.
Suppose that whenever there are n people in the system, then

(i) new arrivals enter the system at an exponential rate λ_n, and
(ii) people leave the system at an exponential rate μ_n.

Such an $\{X(t), t \geq 0\}$ is called a *birth and death process*.

$$0 \underset{\mu_1}{\overset{\lambda_0}{\rightleftarrows}} 1 \underset{\mu_2}{\overset{\lambda_1}{\rightleftarrows}} 2 \underset{\mu_3}{\overset{\lambda_2}{\rightleftarrows}} 3 \ \underset{\cdots}{\overset{\cdots}{\cdots}} \ n-1 \underset{\mu_n}{\overset{\lambda_{n-1}}{\rightleftarrows}} n \underset{\mu_{n+1}}{\overset{\lambda_n}{\rightleftarrows}} n+1 \ \underset{\cdots}{\overset{\cdots}{\cdots}}$$

Suppose the process is at state $i > 0$ at time t. Then

$$B_i = \text{waiting time until the next birth} \sim \text{Exp}(\lambda_i)$$
$$D_i = \text{waiting time until the next death} \sim \text{Exp}(\mu_i)$$

Hence, the waiting time until the next transition out of state i is $\min(B_i, D_i) \sim \text{Exp}(\lambda_i + \mu_i)$, from which we can get

$$\nu_i = \lambda_i + \mu_i, \text{ for } i > 0$$

6.3 Birth and Death Processes (Cont'd)

Moreover, given the process is at state $i > 0$ at time t, the probability that the next transition is a birth rather than a death is

$$P_{i,i+1} = P(B_i < D_i) = \frac{\lambda_i}{\lambda_i + \mu_i},$$

which implies $P_{i,i-1} = P(D_i < B_i) = \frac{\mu_i}{\lambda_i + \mu_i}$, for $i > 0$.

As only birth is possible at state 0, we know $\nu_0 = \lambda_0$ and $P_{01} = 1$.

To sum up, a birth and death process is a CTMC with state space $\mathcal{X} = \{0, 1, 2, \ldots\}$ such that

$$\nu_i = \lambda_i + \mu_i, i > 0, \quad \nu_0 = \lambda_0,$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i},\ P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, i > 0$$

$$P_{01} = 1,\ P_{i,j} = 0 \quad \text{if } |i - j| > 1$$

The parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ are called, respectively, the arrival (or birth) and departure (or death) rates.

Examples of Birth and Death Processes

- Poisson Processes: $\mu_n = 0$, $\lambda_n = \lambda$ for all $n \geq 0$
- Pure Birth Process:

$$\mu_n = 0 \quad \Rightarrow \quad \nu_i = \lambda_i,\ P_{i,i+1} = 1,\ P_{i,i-1} = 0$$

- Yule Processes (Pure Birth Process with Linear Growth rate): If there are n people and each independently gives birth at at an exponential rate λ, then the total rate at which births occur is $n\lambda$.

$$\mu_n = 0, \quad \lambda_n = n\lambda$$

Reason: Let

$B_i =$ time until the ith individual give birth $\sim Exp(\lambda),\ i = 1, \ldots, n$

So the time until the next (first) birth when there are n individuals in the population is

$$\min(B_1, B_2, \ldots, B_n) \sim Exp(\lambda + \lambda + \cdots + \lambda) = Exp(n\lambda)$$

So the rate until the next birth is $\lambda_n = n\lambda$.

Example: Linear Growth Model with Immigration

- each individual independently gives birth at an exponential rate λ
- each individual independently die at at an exponential rate μ
- new immigrants come in at an exponential rate θ

Such a process is a birth-death process with birth and death rates

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Reason: Let

$B_i =$ time until the ith individual give birth $\sim Exp(\lambda),\ i = 1, \ldots, n$
$T =$ time until the next new immigrant comes in $\sim Exp(\theta)$

So the time until the population size increase from n to $n+1$ is

$$\min(B_1, \ldots, B_n, T) \sim Exp(\lambda + \cdots + \lambda + \theta) = Exp(n\lambda + \theta)$$

So the rate until the next birth is $\lambda_n = n\lambda + \theta$.

Similarly, one can show that the death rate is $\mu_n = n\mu$.

Example: $M/M/s$ Queueing Model

- s servers
- Poisson arrival of customers, rate $= \lambda$
- Exponential service time, rate $= \mu$

$\Rightarrow$ a birth and death process with constant birth rate $\lambda_n = \lambda$, and death (departure)rate $\mu_n = \min(n, s)\mu$.

Reason: Suppose, there are n customer in the system at time t. At most $\min(n, s)$ of them are being served. Let S_i be remaining service time of the ith server $\sim$ Exp(μ). Then, the waiting time until the next departure is

$$\min(S_1, \ldots, S_{\min(s,n)}) \sim \text{Exp}(\min(s, n)\mu).$$

6.4 The Transition Probability Function $P_{ij}(t)$

Recall the transition probability function $P_{ij}(t)$ of a CTMC $\{X(t), t \geq 0\}$ is

$$P_{ij}(t) = \mathrm{P}(X(t+s) = j | X(s) = i)$$

Example. (Poisson Processes with rate λ)

$$\begin{aligned} P_{ij}(t) &= \mathrm{P}(N(t+s) = j | N(s) = i) \\ &= \mathrm{P}(N(t+s) - N(s) = j - i) = \begin{cases} e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases} \end{aligned}$$

Properties of Transition Probability Functions

- $P_{ij}(t) \geq 0$ for all $i, j \in \mathcal{X}$ and $t \geq 0$
- (Row sums are 1) $\sum_j P_{ij}(t) = 1$ for all $i \in \mathcal{X}$ and $t \geq 0$

Lemma 6.3 Chapman-Kolmogorov Equation

For all $i, j \in \mathcal{X}$ and $t \geq 0$,

$$P_{ij}(t+s) = \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(s)$$

Proof.

$$\begin{aligned}
&P_{ij}(t+s) \\
&= \mathrm{P}(X(t+s) = j | X(0) = i) \\
&= \sum_{k \in \mathcal{X}} \mathrm{P}(X(t+s) = j, X(t) = k | X(0) = i) \\
&= \sum_{k \in \mathcal{X}} \mathrm{P}(X(t+s) = j | X(t) = k, X(0) = i) \mathrm{P}(X(t) = k | X(0) = i) \\
&= \sum_{k \in \mathcal{X}} \mathrm{P}(X(t+s) = j | X(t) = k) \mathrm{P}(X(t) = k | X(0) = i) \text{ (Markov Property)} \\
&= \sum_{k \in \mathcal{X}} P_{kj}(s) P_{ik}(t)
\end{aligned}$$

The matrix notation

Let $\mathbf{P}(t) = [P_{ij}(t)]$ be the transition matrix at time t.
We have $\mathbf{P}(0) = \mathbf{I}$. And C-K equations read

$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s)$$

One way to specify a CTMC is through $\{\mathbf{P}(t)\}_{t \geq 0}$. But this requires an infinite number of matrices. Can we simplify it?

Key: use derivatives $\mathbf{P}'(t)$

Transition rate matrix / infinitesimal generator $\mathbf{Q}$

Assume that

$$\mathbf{P}'(0) = \lim_{h \to 0} \frac{\mathbf{P}(h) - \mathbf{P}(0)}{h} \quad \text{exists.}$$

In other words, for each i, j,

$$P'_{ij}(0) = \lim_{h \to 0} \frac{P_{ij}(h) - P_{ij}(0)}{h} \quad \text{exists.}$$

We will denote such limit as $\mathbf{Q} = [q_{ij}]$, the transition rate matrix.
How about $\mathsf{P}'(t)$ for $t > 0$?

Kolmogorov's equations

By definition, one has

$$\mathsf{P}'(t) = \lim_{h\to 0} \frac{\mathbf{P}(t+h) - \mathbf{P}(t)}{h} = \lim_{h\to 0} \frac{\mathbf{P}(t)\mathbf{P}(h) - \mathbf{P}(t)}{h}$$
$$= \lim_{h\to 0} \frac{\mathbf{P}(t)(\mathbf{P}(h) - \mathbf{I})}{h} = \mathbf{P}(t)\mathbf{Q}.$$

This is the so-called Kolmogorov's forward equations.

Similarly you can prove backward equations

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t).$$

These imply $\mathbf{P}(t) = \exp(t\mathbf{Q})$.

Transition rate matrix

How to compute $\mathbf{Q}$?

Lemma 6.2a

For any $i, j \in \mathcal{X}$, we have

$$q_{ii} = \lim_{h \to 0} \frac{P_{ii}(h) - 1}{h} = -\nu_i$$

Proof. Let T_i be the amount of time the process stays in state i before moving to other states.

$$\begin{aligned}
P_{ii}(h) &= \mathrm{P}(X(h) = i | X(0) = i) \\
&= \mathrm{P}(X(h) = i, \text{no transition in (0,h]} | X(0) = i) \\
&\quad + \mathrm{P}(X(h) = i, \text{2 or more transition in (0,h]} | X(0) = i) \\
&= \mathrm{P}(T_i > h) + o(h) \\
&= e^{-\nu_i h} + o(h) \\
&= 1 - \nu_i h + o(h)
\end{aligned}$$

Lemma 6.2b

For any $i \neq j \in \mathcal{X}$, we have

$$q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h)}{h} = \nu_i P_{ij}$$

Proof.

$$\begin{aligned}
P_{ij}(h) &= \mathrm{P}(X(h) = j | X(0) = i) \\
&= \mathrm{P}(X(h) = j, 1 \text{ transition in } (0,h] | X(0) = i) \\
&\quad + \mathrm{P}(X(h) = j, 2 \text{ or more transition in } (0,h] | X(0) = i) \\
&= \mathrm{P}(T_i < h) P_{ij} + o(h) \\
&= (1 - e^{-\nu_i h}) P_{ij} + o(h) \\
&= \nu_i P_{ij} h + o(h)
\end{aligned}$$

For finite state space case $\mathcal{X} = \{1, 2, \ldots, m\}$, define the matrices

$$\mathbf{P}(t) = \begin{bmatrix} P_{11}(t) & \cdots & P_{1m}(t) \\ \vdots & & \vdots \\ P_{m1}(t) & \cdots & P_{mm}(t) \end{bmatrix}, \quad \mathbf{P}'(t) = \begin{bmatrix} P'_{11}(t) & \cdots & P'_{1m}(t) \\ \vdots & & \vdots \\ P'_{m1}(t) & \cdots & P'_{mm}(t) \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & & \vdots \\ q_{m1} & \cdots & q_{mm} \end{bmatrix} = \begin{bmatrix} -\nu_1 & \nu_1 P_{12} & \cdots & \nu_1 P_{1m} \\ \nu_2 P_{21} & -\nu_2 & \cdots & \nu_2 P_{2m} \\ \vdots & \vdots & & \vdots \\ \nu_m P_{m1} & \nu_m P_{m2} & \cdots & -\nu_m \end{bmatrix}$$

In matrix notation,
Forward equation: $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$
Backward equation: $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$