STAT253/317 Lecture 11

6.5. Limiting Probabilities

Definition. Just like discrete-time Markov chains, if the probability that a continuous-time Markov chain will be in state j at time t, $P_{ij}(t)$, converges to a limiting value P_j independent of the initial state i, for all $i \in \mathcal{X}$

$$P_j = \lim_{t\to\infty} P_{ij}(t) > 0$$

then we say P_j is the *limiting probability* of state j. If P_j exists for all $j \in \mathcal{X}$, we say $\{P_j\}_{j\in\mathcal{X}}$ is the *limiting distribution* of the process.

Remark. If $\lim_{t\to\infty} P_{ij}(t)$ exists, we must have

$$\lim_{t\to\infty} P'_{ij}(t) = 0.$$

Balanced equations

Recall the forward equations $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

If you set $t \to \infty$, you have

$$0 = p^\top \mathbf{Q},$$

where $p = (P_1, P_2, \ldots)^\top$
This is the same as saying that

$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} \quad \text{for all } j \in \mathcal{X}$$

Interpretation of the Balanced Equations

$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$$

$$\nu_j P_j = \text{rate at which the process } \textbf{leaves} \text{ state } j$$

$$\sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} = \text{rate at which the process } \textbf{enters} \text{ state } j$$

Balanced equations means that the rates at which the process enters and leaves state j are equal.

The limiting distribution $\{P_j\}_{j \in \mathcal{X}}$ can be obtained by solving the balanced equations along with the equation $\sum_{j \in \mathcal{X}} P_j = 1$.

Remarks. Just like discrete-time Markov chains, a sufficient condition for the existence of a limiting distribution is that the chain is irreducible and positive recurrent.

Examples

- **Poisson processes**: $\mu_n = 0$, $\lambda_n = \lambda$ for all $n \geq 0$

  $$\nu_i = \lambda,\ P_{i,i+1} = 1, \quad q_{i,i+1} = \nu_i P_{i,i+1} = \lambda$$

  Balanced equations:

  $$\nu_j P_j = P_{j-1} q_{j-1,j} \quad \Rightarrow \quad \lambda P_j = \lambda P_{j-1} \quad \Rightarrow \quad P_j = P_{j-1}$$

  No limiting distribution exists. The chain is not irreducible. All states are transient.

- **Pure birth processes with $\lambda_n > 0$ for all n**
  No limiting distribution exists. All states are transient.

Birth and Death Processes

For a birth and death process,

$$
\begin{aligned}
\nu_0 &= \lambda_0, \\
\nu_i &= \lambda_i + \mu_i,\ \ i > 0 \\
P_{01} &= 1, \\
P_{i,i+1} &= \tfrac{\lambda_i}{\lambda_i+\mu_i}, \quad i > 0 \\
P_{i,i-1} &= \tfrac{\mu_i}{\lambda_i+\mu_i}, \quad i > 0 \\
P_{i,j} &= 0 \qquad\quad \text{if } |i-j| > 1
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
q_{i,i+1} &= \nu_i P_{i,i+1} = \lambda_i,\ \ i \geq 0 \\
q_{i,i-1} &= \nu_i P_{i,i-1} = \mu_i, i \geq 1
\end{aligned}
$$

Balanced Equations for Birth and Death Processes

The balanced equations $\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$ for a birth and death process are

$$
\begin{aligned}
\lambda_0 P_0 &= \mu_1 P_1 \\
(\mu_1 + \lambda_1) P_1 &= \lambda_0 P_0 + \mu_2 P_2, \\
(\mu_2 + \lambda_2) P_2 &= \lambda_1 P_1 + \mu_3 P_3, \\
&\vdots \\
(\mu_{n-1} + \lambda_{n-1}) P_{n-1} &= \lambda_{n-2} P_{n-2} + \mu_n P_n \\
(\mu_n + \lambda_n) P_n &= \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}
\end{aligned}
$$

Adding up all the equations above and eliminating the common terms on both sides, we get

$$\lambda_n P_n = \mu_{n+1} P_{n+1} \quad n \geq 0,$$

We hence just need to solve $\lambda_n P_n = \mu_{n+1} P_{n+1}$ for the limiting distribution.

6.6. Time Reversibility

Definition. A continuous-time Markov chain with state space $\mathcal{X}$ is *time reversible* if

$$P_i q_{ij} = P_j q_{ji}, \quad \text{for all } i, j \in \mathcal{X} \quad \text{(detailed balanced equation)}$$

If a distribution $\{P_j\}$ on $\mathcal{X}$ satisfies the detailed balanced equation, then it is a stationary distribution for the process.

Example. We have just shown that for Birth and Death processes, the balanced equations would lead to the detailed balanced equations, which are

$$\lambda_n P_n = \mu_{n+1} P_{n+1}, \quad n \geq 0$$

Limiting Dist'n for Birth and Death Processes

Solving $\lambda_n P_n = \mu_{n+1} P_{n+1}$, $n \geq 0$ for the limiting distribution, we get

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1}}{\mu_n} \frac{\lambda_{n-2}}{\mu_{n-1}} P_{n-2} = \ldots = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} P_0$$

To meet the requirement $\sum_{n=0}^{\infty} P_n = 1$, we need

$$\sum_{n=0}^{\infty} P_n = P_0 + P_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} = 1$$

In other words, to have a limiting distribution, it is necessary that

$$\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} < \infty$$

Limiting Dist'n for Birth and Death Processes (Cont'd)

If $\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}$ is finite, the limiting distribution is

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}}$$

and

$$P_k = \frac{\lambda_0\lambda_1\cdots\lambda_{k-1}/(\mu_1\mu_2\cdots\mu_k)}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}}, \quad k \geq 1$$

Lemma: (Ratio Test) If $a_n \geq 0$ for all n, then

$$\sum_{n=1}^{\infty} a_n \begin{cases} < \infty & \text{if } \lim_{n\to\infty} a_n/a_{n-1} < 1 \\ = \infty & \text{if } \lim_{n\to\infty} a_n/a_{n-1} > 1 \end{cases}$$

For $a_n = \dfrac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}$, $a_n/a_{n-1} = \lambda_{n-1}/\mu_n$. By the ratio test, if

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\mu_n} < 1,$$

then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} < \infty$, the limiting distribution exists.

Example 6.4 Linear Growth Model with Immigration

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Using the Ratio Test,

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\mu_n} = \lim_{n\to\infty} \frac{(n-1)\lambda + \theta}{n\mu} = \frac{\lambda}{\mu}$$

So the linear growth model has a limiting distribution if $\lambda < \mu$.

Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

- single-server service station. Service times are i.i.d. $\sim Exp(\mu)$
- Poisson arrival of customers with rate λ
- Upon arrival, a customer would
  - go into service if the server is free (queue length $= 0$)
  - join the queue if 1 to $N-1$ customers in the station, or
  - walk away if N or more customers in the station

Q: What fraction of potential customers are lost?

Let $X(t)$ be the number of customers in the station at time t.

$\{X(t),\ t \geq 0\}$ is a birth-death process with the birth and death rates below

$$\mu_n = \begin{cases} 0 & \text{if } n = 0 \\ \mu & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad \lambda_n = \begin{cases} \lambda & \text{if } 0 \leq n < N \\ 0 & \text{if } n \geq N \end{cases}$$

Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

Solving $\lambda_n P_n = \mu_{n+1} P_{n+1}$ for the limiting distribution

$$
\begin{aligned}
P_1 &= (\lambda/\mu) P_0 \\
P_2 &= (\lambda/\mu) P_1 = (\lambda/\mu)^2 P_0 \\
&\vdots \\
P_i &= (\lambda/\mu)^i P_0, \qquad\qquad i = 1, 2, \ldots, N
\end{aligned}
$$

Plugging $P_i = (\lambda/\mu)^i P_0$ into $\sum_{i=0}^{N} P_i = 1$, one can solve for P_0 and get

$$P_i = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^i$$

Answer: The fraction of customers lost is $P_N = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} (\lambda/\mu)^N$

Duration Times for Birth and Death Processes

Let

$$T_i = \text{time to move from state } i \text{ to state } i+1, \quad i = 0, 1, \ldots.$$

Suppose at some moment $X(t) = i$. Let

$$B_i = \text{time until the next birth} \sim Exp(\lambda_i)$$
$$D_i = \text{time until the next death} \sim Exp(\mu_i)$$

Then

$$
\begin{aligned}
T_i &= \begin{cases} B_i & \text{if the next step is } i \to i+1, \text{ i.e., } B_i < D_i \\ D_i + T_{i-1} + T_i^* & \text{if the next step is } i \to i-1, \text{ i.e., } D_i < B_i \end{cases} \\
&= \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i \text{, occur w/ prob. } \frac{\lambda_i}{\lambda_i + \mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i \text{, occur w/ prob. } \frac{\mu_i}{\lambda_i + \mu_i} \end{cases}
\end{aligned}
$$

Note

- T_i^* has the same distribution as T_i
- T_{i-1} and T_i^* are indep. of B_i and D_i because it's Markov

Duration Times for Birth and Death Processes

Taking expected value of

$$T_i = \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i \text{, occur w/ prob. } \frac{\lambda_i}{\lambda_i+\mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i \text{, occur w/ prob. } \frac{\mu_i}{\lambda_i+\mu_i} \end{cases}$$

we get

$$\begin{aligned} \mathbb{E}[T_i] &= \mathbb{E}[\min(B_i, D_i)] + (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i]) \frac{\mu_i}{\lambda_i + \mu_i} \\ &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i]) \end{aligned}$$

We obtain the recursive formula

$$\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$$

Duration Times for Birth and Death Processes (Cont'd)

Since $T_0 \sim Exp(\lambda_0)$, $\mathbb{E}[T_0] = 1/\lambda_0$.
Using the recursive formula $\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$, we have

$$\begin{aligned}
\mathbb{E}[T_0] &= \frac{1}{\lambda_0} \\
\mathbb{E}[T_1] &= \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\mathbb{E}[T_0] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0} \\
\mathbb{E}[T_2] &= \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0}\right) = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2 \lambda_1} + \frac{\mu_2 \mu_1}{\lambda_2 \lambda_1 \lambda_0} \\
&\vdots \\
\mathbb{E}[T_i] &= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}\mathbb{E}[T_{i-1}] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \cdots + \frac{\mu_i \mu_{i-1} \cdots \mu_2 \mu_1}{\lambda_i \lambda_{i-1} \cdots \lambda_2 \lambda_1 \lambda_0} \\
&= \frac{1}{\lambda_i}\left(1 + \sum_{k=1}^{i} \prod_{j=1}^{k} \frac{\mu_{i-j+1}}{\lambda_{i-j}}\right)
\end{aligned}$$