STAT253/317 Lecture 12

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Chapter 7 Renewal Processes

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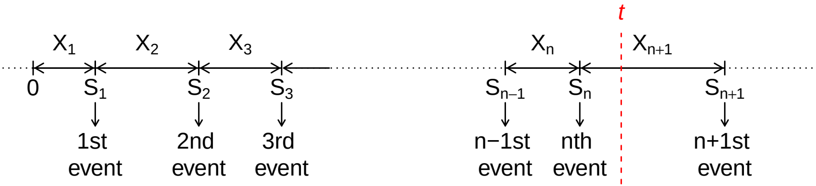
Recall the interarrival times of a Poisson process are i.i.d exponential random variables.

A **renewal process** is a counting process of which the interarrival times are i.i.d., but may not have an exponential distribution.

Definition of a Renewal Process



Let X_1, $X_2, \ldots$ be i.i.d random variables with $\mathbb{E}[X_i] < \infty$, and $\mathrm{P}(X_i = 0) < 1$. Let

$$S_0 = 0, \quad S_n = X_1 + \ldots + X_n,\ n \geq 1.$$

Define

$$N(t) = \max\{n : S_n \leq t\}.$$

Then $\{N(t), t \geq 0\}$ is called a *renewal process*.

- Events are called "*renewals*". The interarrival times between events X_1, $X_2, \ldots$ are also called "renewals"
- A more general definition allows the first renewal X_1 to be of a different distribution, called a **delayed renewal process**

Renewal Processes Are Well-Defined

Renewal processes are well-defined in the sense that

$$P(\max\{n : S_n \leq t\} < \infty) = 1 \quad \text{for all } t > 0.$$

By SLLN $\Rightarrow P\left(\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1]\right) = 1$

$\Rightarrow \; P\left(\lim_{n\to\infty} S_n = \infty\right) = 1$

$\Rightarrow$ For any t, w/ prob. 1 $S_n < t$ for only finitely many n

$\Rightarrow \; P(\max\{n : S_n \leq t\} < \infty) = 1$ for all $t > 0$

Examples of Renewal Processes

- Replacement of light bulbs: $N(t) = \#$ of replaced light bulbs by time t, is a renewal process
- Consider a homogeneous, irreducible, positive recurrent, discrete time Markov chain, started from a state i. Let

$$N_i(t) = \text{number of visits to state } i \text{ by time } t.$$

Then $\{N_i(t), t \geq 0\}$ is a renewal process.

Basic Properties of Renewal Processes

- $\mathrm{P}(\lim_{t\to\infty} N(t) = \infty) = 1$

  Reason: $\lim_{t\to\infty} N(t) < \infty$ can happen only when $X_i = \infty$ for some i.

  $$\left\{\lim_{t\to\infty} N(t) < \infty\right\} \subseteq \bigcup\nolimits_{i=1}^{\infty} \{X_i = \infty\}$$

  However, as the interarrival times of a renewal process are required to have finite means $\mathbb{E}[X_i] < \infty$, which implies $\mathrm{P}(X_i = \infty) = 0$, we must have

  $$\mathrm{P}\left(\lim_{t\to\infty} N(t) < \infty\right) \le \mathrm{P}\left(\bigcup_{i=1}^{\infty} \{X_i = \infty\}\right) \le \sum_{i=1}^{\infty} \mathrm{P}(X_i = \infty) = 0.$$

- Not memoryless in general

  $\Rightarrow$ No independent or stationary increments in general
  $\mathrm{P}(N(t+h) - N(t) = 1)$ depends on the current lifetime $A(t) = t - S_{N(t)}$

Things of Interest

- Distribution of $N(t)$:

$$\mathrm{P}(N(t) = n), \quad n = 0, 1, 2, \dots$$

- Renewal function:

$$m(t) = \mathbb{E}[N(t)]$$

- Residual life (a.k.a. excess life, overshoot, excess over the boundary):

$$B(t) = S_{N(t)+1} - t$$

- Current age (a.k.a. current life, undershoot):

$$A(t) = t - S_{N(t)}$$

- Total life: $C(t) = A(t) + B(t)$
- Inspection paradox: $C(t)$ and the interarrival time X_i have different distributions.

7.2. Distribution of $N(t)$

Let

$$F_n(t) = \mathrm{P}(S_n \leq t)$$

be the CDF of the arrival time $S_n = X_1 + \cdots + X_n$ of the nth event. Observe that

$$\{N(t) \geq n\} \quad \Leftrightarrow \quad \{S_n \leq t\}$$

Thus

$$\begin{aligned} \mathrm{P}(N(t) = n) &= \mathrm{P}(N(t) \geq n) - \mathrm{P}(N(t) \geq n+1) \\ &= \mathrm{P}(S_n \leq t) - \mathrm{P}(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

This formula looks simple but is generally <u>USELESS</u> in practice since $F_n(t)$ is often intractable.

The Renewal Function $m(t)$

Recall that if a random variable X takes non-negative integer values $\{0, 1, 2, \ldots\}$, then $\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathrm{P}(X \geq n)$. Therefore the renewal function can be written as

$$\begin{aligned} m(t) = \mathbb{E}[N(t)] &= \sum_{n=1}^{\infty} \mathrm{P}(N(t) \geq n) \\ &= \sum_{n=1}^{\infty} \mathrm{P}(S_n \leq t) = \sum_{n=1}^{\infty} F_n(t) \end{aligned}$$

- It can be shown that the renewal function $m(t)$ can uniquely determine the interarrival distribution F. So the only renewal process with linear renewal function $m(t) = \lambda t$ is the Poisson process with rate λ.
- The formula $m(t) = \sum_{n=1}^{\infty} F_n(t)$ is again generally useless since $F_n(t)$ often times has no closed form expression. We need more tools.

The Renewal Equation

Conditioning on $X_1 = x$, observe that

$$(N(t)|X_1 = x) = \begin{cases} 1 + N(t-x) & \text{if } x \leq t \\ 0 & \text{if } x > t \end{cases}$$

Assuming that the interarrival distribution F is continuous with density function f. Then

$$\begin{aligned} m(t) = \mathbb{E}[N(t)] &= \int_0^\infty \mathbb{E}[N(t)|X_1 = x] f(x) dx \\ &= \int_0^t (1 + \mathbb{E}[N(t-x)]) f(x) dx + \int_t^\infty 0 f(x) dx \\ &= \int_0^t (1 + m(t-x)) f(x) dx = F(t) + \int_0^t m(t-x) f(x) dx \end{aligned}$$

The equation

$$m(t) = F(t) + \int_0^t m(t-x) f(x) dx$$

is called the *renewal equation*.

Example 7.3

Suppose the interarrival times X_i are i.i.d. uniform on $(0, 1)$. The density and CDF of X_i's are respectively

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

For $0 \leq t \leq 1$, the renewal equation is

$$m(t) = t + \int_0^t m(t-x)dx = t + \int_0^t m(x)dx$$

Differentiating the equation with respect to t yields

$$m'(t) = 1 + m(t) \Rightarrow \frac{d}{dt}(1 + m(t)) = 1 + m(t) \Rightarrow 1 + m(t) = Ke^t.$$

or $m(t) = Ke^t - 1$. Since $m(0) = 0$, we can see that $K = 1$ and obtain that $m(t) = e^t - 1$ for $0 \leq t \leq 1$.

What if $1 \leq t \leq 2$?

For $1 \le t \le 2$, $F(t) = 1$, the renewal equation is

$$m(t) = 1 + \int_0^1 m(t-x)dx = 1 + \int_{t-1}^{t} m(x)dx$$

Differentiating the preceding equation yields

$$m'(t) = m(t) - m(t-1) = m(t) - [e^{t-1} - 1] = m(t) + 1 - e^{t-1}$$

Multiplying both side by e^{-t}, we get

$$\underbrace{e^{-t}(m'(t) - m(t))}_{\frac{d}{dt}[e^{-t}m(t)]} = e^{-t} - e^{-1}$$

Integrating over t from 1 to t, we get

$$\begin{aligned} e^{-t}m(t) &= e^{-1}m(1) + e^{-1}\int_1^t e^{-(s-1)} - 1ds \\ &= e^{-1}m(1) + e^{-1}[1 - e^{-(t-1)} - (t-1)] \\ \Rightarrow m(t) &= e^{t-1}m(1) + e^{t-1} - 1 - e^{t-1}(t-1) \\ &= e^{t} + e^{t-1} - 1 - te^{t-1} \qquad (\text{Note } m(1) = e-1) \end{aligned}$$

In general for $n \leq t \leq n+1$, the renewal equation is

$$m(t) = 1 + \int_{t-1}^{t} m(x)dx \quad \Rightarrow \quad m'(t) = m(t) - m(t-1)$$

Multiplying both side by e^{-t}, we get

$$\frac{d}{dt}(e^{-t}m(t)) = e^{-t}(m'(t) - m(t)) = -e^{-t}m(t-1)$$

Integrating over t from 1 to t, we get

$$e^{-t}m(t) = e^{-n}m(n) - \int_{n}^{t} e^{-s}m(s-1)ds$$

Thus we can find $m(t)$ iteratively.

7.3. Limit Theorems

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d interarrival times X_i, $i = 1, 2, \ldots$ and $\mathbb{E}[X_i] = \mu$.

Explicit forms of $N(t)$ and $m(t) = \mathbb{E}[N(t)]$ are usually *unavailable*. However the limiting behavior of $N(t)$ and $m(t)$ is useful and intuitively makes sense.

As $t \to \infty$,

- $\dfrac{N(t)}{t} \to \dfrac{1}{\mu}$ with probability 1 **(Proposition 7.1)**
- $\dfrac{m(t)}{t} \to \dfrac{1}{\mu}$ **(Thm 7.1 Elementary Renewal Theorem)**

Remark.

- The number $1/\mu$ is called the **rate** of the renewal process
- Theorem 7.1 is not a simple consequence of Proposition. 7.1, since $X_n \to X$ w/ prob. 1 does not ensure $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

$X_n \to X$ Does Not Ensure $\mathbb{E}[X_n] \to \mathbb{E}[X]$

Example 7.8 Let U be a random variable which is uniformly distributed on (0, 1); and define the random variables X_n, $n \geq 1$, by

$$X_n = \begin{cases} 0 & \text{if } U > 1/n \\ n & \text{if } U \leq 1/n \end{cases}$$

Then $\mathrm{P}(X_n = 0) = P(U > 1/n) = 1 - 1/n \to 1$ as $n \to \infty$. So with probability 1

$$X_n \to X = 0.$$

However,

$$\mathbb{E}[X_n] = 0\mathrm{P}(X_n = 0) + n\mathrm{P}(X_n = n) = n \times \frac{1}{n} = 1 \quad \text{for all } n \geq 1.$$

and hence $\lim_{n\to\infty} \mathbb{E}[X_n] = 1 \neq \mathbb{E}[X] = \mathbb{E}[0] = 0.$

Example 7.6 (M/G/1 with no Queue)

- Single-server bank
- Potential customers arrive at a Poisson rate λ
- Customers enter the bank only if the server is free
- Service times are i.i.d. with mean μ_G, indep. of the arrival
- Let $N(t) =$ number of customers entry the bank by time t and those who arrive finding the server busy and walk away don't count. Is $\{N(t) : t \geq 0\}$ a (delayed) renewal process?

Ans. An interarrival time $T_i = G_i + W_i$ where

$G_i =$ service time, i.i.d., w/ mean μ_G

$W_i =$ waiting time until the next customer arrives after the previous one i

As potential customers arrive following a Poisson process, by the memoryless property, W_i's are i.i.d. Exp(λ).

The interarrival times $\{T_i\} = \{G_i + W_i\}$ are i.i.d. The events of customers entering constitutes a renewal process

Example 7.6 (M/G/1 with no Queue)

Q: What is the rate at which customers enter the bank?

- As $\mathbb{E}[T_i] = \mathbb{E}[G_i] + \mathbb{E}[W_i] = \mu_G + \frac{1}{\lambda}$, by the Elementary Renewal Theorem, the rate is

$$\frac{1}{\mathbb{E}[T_i]} = \frac{1}{\mu_G + \frac{1}{\lambda}} = \frac{\lambda}{\lambda\mu_G + 1}$$

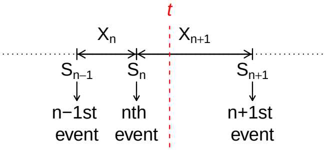
Q: What is the proportion of potential customers that are lost?

- As potential customers arrive at rate λ, and customers enter at the rate $\frac{\lambda}{\lambda\mu_G + 1}$, the proportion that actually enter the bank is

$$\frac{\lambda/(\lambda\mu_G + 1)}{\lambda} = \frac{1}{\lambda\mu_G + 1}$$

So the proportion that is lost is $1 - \frac{1}{\lambda\mu_G + 1} = \frac{\lambda\mu_G}{\lambda\mu_G + 1}$.

Proof of Proposition 7.1



Since $S_{N(t)} \leq t < S_{N(t)+1}$, we know

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}.$$

By SLLN, $\frac{S_{N(t)}}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \to \mu$ as $N(t) \to \infty$, we obtain $\frac{S_{N(t)}}{N(t)} \to \mu$ as $t \to \infty$. Furthermore, writing

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \times \frac{N(t)+1}{N(t)}$$

we have that $S_{N(t)+1}/(N(t)+1) \to \mu$ by the same reasoning as before and

$$\frac{N(t)+1}{N(t)} \to 1 \text{ as } t \to \infty \quad \text{since } P(\lim_{t\to\infty} N(t) = \infty) = 1$$

Hence, $S_{N(t)+1}/N(t) \to \mu$.

Stopping Time

Definition. Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables. An integer-valued random variable $N > 0$ is said to be a *stopping time* w/ respect to $\{X_n : n \geq 1\}$ if the event $\{N = n\}$ is independent of $\{X_k : k \geq n+1\}$.

Example. (*Independent case.*)
If N is independent of $\{X_n : n \geq 1\}$, then N is a stopping time.

Example. (*Hitting Time* I.) For any set A, the first time X_n hits set A, $N_A = \min\{n : X_n \in A\}$, is a stopping time because

$$\{N_A = n\} = \{X_i \notin A \text{ for } i = 1, 2, \ldots, n-1, \text{ but } X_n \in A\}$$

is independent of $\{X_k : k \geq n+1\}$.

Example. (*Hitting Time* II.) For $n \geq 1$, let $S_n = \sum_{k=1}^{n} X_k$.
For any set A, $N_A = \min\{n : S_n \in A\}$, the first time S_n hits set A, is also a stopping time w/ respect to $\{X_n : n \geq 1\}$ because
$\{N_A = n\} = \{\sum_{k=1}^{i} X_k \notin A \text{ for } 1 \leq i \leq n-1, \text{ but } \sum_{k=1}^{n} X_k \in A\}$

is independent of $\{X_k : k \geq n+1\}$.

Example of Non-Stopping Times

- (*Last visit time*) The last time that X_n visit a set A

  $$N_A = \max\{n : X_n \in A\}$$

  is NOT a stopping time.
  Clearly we need to know whether A will be visited again in the future to determine such a time.

- The time X_n reaches its maximum,

  $$N = \min\{n : X_n = \max_{k \geq 1} X_k\},$$

  is NOT a stopping time since

  $$\{N = n\} = \{X_n > X_k \text{ for } 1 \leq k < n \text{ and } k \geq n+1\}$$

  depends on $\{X_k : k \geq n+1\}$.

Renewal Processes and Stopping Times

Consider a renewal process $N(t)$. With respect to its interarrival times X_1, $X_2, \ldots$,

- $N(t)$ is NOT a stopping time.

  $$N(t) = n \Leftrightarrow X_1 + \cdots + X_n \leq t \text{ and } X_1 + \cdots + X_{n+1} > t,$$

  depends on X_{n+1}.
- But $N(t) + 1$ is a stopping time, since

  $$\begin{aligned} N(t) + 1 = n &\Leftrightarrow N(t) = n - 1 \\ &\Leftrightarrow X_1 + \cdots + X_{n-1} \leq t \text{ and } X_1 + \cdots + X_n > t, \end{aligned}$$

  is independent of X_{n+1}, $X_{n+2}, \ldots$.

Wald's Equation

If $X_1, X_2 \ldots$ are i.i.d. with $\mathbb{E}[X_i] < \infty$, and if N is a stopping time for this sequence with $\mathbb{E}[N] < \infty$, then

$$\mathbb{E}\left[\sum_{j=1}^{N} X_j\right] = \mathbb{E}[N]\mathbb{E}[X_1]$$

Proof. Let us define the indicator variable

$$I_j = \begin{cases} 1 & \text{if } j \leq N \\ 0 & \text{if } j > N. \end{cases}$$

We have

$$\sum_{j=1}^{N} X_j = \sum_{j=1}^{\infty} X_j I_j$$

Hence

$$\mathbb{E}\left[\sum_{j=1}^{N} X_j\right] = \mathbb{E}\left[\sum_{j=1}^{\infty} X_j I_j\right] = \sum_{j=1}^{\infty} \mathbb{E}[X_j I_j] \qquad (1)$$

Proof of Wald's Equation (Cont'd)

Note I_j and X_j are independent because

$$I_j = 0 \quad \Leftrightarrow \quad N < j \quad \Leftrightarrow \quad N \leq j - 1$$

and the event $\{N \leq j-1\}$ depends on $X_1, \ldots, X_{j-1}$ only, but not X_j. From (1), we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{N} X_j\right] &= \sum_{j=1}^{\infty} \mathbb{E}[X_j I_j] = \sum_{j=1}^{\infty} \mathbb{E}[X_j]\mathbb{E}[I_j] \\
&= \mathbb{E}[X_1] \sum_{j=1}^{\infty} \mathbb{E}[I_j] = \mathbb{E}[X_1] \sum_{j=1}^{\infty} \mathrm{P}(N \geq j) \\
&= \mathbb{E}[X_1]\mathbb{E}[N]
\end{aligned}$$

Here we use the alternative formula $\mathbb{E}[N] = \sum_{j=1}^{\infty} \mathrm{P}(N \geq j)$ to find expected values of non-negative integer valued random variables.

Proposition 7.2

$$\mathbb{E}[S_{N(t)+1}] = \mu(m(t)+1)$$

Proof: Since $N(t)+1$ is a stopping time, by Wald's equation, we have

$$\mathbb{E}[S_{N(t)+1}] = \mathbb{E}\left[\sum_{j=1}^{N(t)+1} X_j\right] = \mathbb{E}[N(t)+1]\mathbb{E}[X_1] = (m(t)+1)\mu$$

Since $S_{N(t)+1} = t + Y(t)$, where $Y(t)$ is the residual life at t, taking expectations and using the result above yields

$$\mathbb{E}[S_{N(t)+1}] = \mu(m(t)+1) = t + \mathbb{E}[Y(t)].$$

So far we have proved Proposition 7.2 and can deduce that

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{\mathbb{E}[Y(t)]}{t\mu}.$$

Proof of the Elementary Renewal Theorem

First from Proposition 7.2, we have

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{\mathbb{E}[Y(t)]}{t\mu} \geq \frac{1}{\mu} - \frac{1}{t} \quad \Rightarrow \quad \lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

It remains to show that $\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$.

If the interarrival times $X_1, X_2, \ldots$ are bounded by a constant M, then the residual life $Y(t)$ is also bounded by M. Hence,

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \lim_{t\to\infty} \frac{1}{\mu} - \frac{1}{t} + \frac{M}{t\mu} = \frac{1}{\mu}$$

The Elementary Renewal Theorem for renewal process with **bounded interarrival times** is proved.

Proof of the Elementary Renewal Theorem (Cont'd)

In general, if the interarrival times $X_1, X_2, \ldots$ are not bounded, we fix a constant M and define a new renewal process $N_M(t)$ with the truncated interarrival times

$$\min(X_1, M), \min(X_2, M), \ldots, \min(X_n, M), \ldots.$$

Because $\min(X_i, M) \leq X_i$ for all i, it follows that $N_M(t) \geq N(t)$ for all t.

$$\lim_{t\to\infty} \frac{m(t)}{t} = \lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} \leq \lim_{t\to\infty} \frac{\mathbb{E}[N_M(t)]}{t} = \frac{1}{\mathbb{E}[\min(X_1, M)]}$$

by the Elementary Renewal Theorem with bounded interarrival times. Note the inequality above is valid for all $M > 0$. Letting $M \to \infty$ yields

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}.$$

Here we use the fact that $\mathbb{E}[\min(X_1, M)] \to \mathbb{E}[X_1] = \mu$ as $M \to \infty$.