STAT253/317 Lecture 4: Limiting theorems for Markov chains

Recall the two questions

1. As time goes to infinity, does the fraction of time spent in a given state converge? Mathematically, we aim to study

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} 1\{X_i = k\}}{n}.$$

2. As time goes to infinity, does the probability of being in a given state converge to a limit? This is given by

$$\lim_{n\to\infty} P_{ij}^{(n)}.$$

Positive Recurrence and Null Recurrence

For a Markov chain, define the first return time to a state i

$$T_i = \min\{n > 0 : X_n = i \mid X_0 = i\}$$

We say a state i is

- **positive recurrent** if i is recurrent and $\mathbb{E}[T_i] < \infty$.
- **null recurrent** if i is recurrent but $\mathbb{E}[T_i] = \infty$.

We say a state is **ergodic** if it is aperiodic and positive recurrent. Positive Recurrence is a Class Property. Similarly, Null Recurrence is a Class Property.

The Fundamental Limit Theorem of Markov Chain II

For an **irreducible** Markov chain, it is **positive recurrent** if and only if there exists a stationary distribution, i.e., a solution to the set of equations:

$$\pi_i \geq 0, \quad \sum_{i \in \mathfrak{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$$

Moreover, if a solution exists then it is unique, and is given by

$$\pi_j = \frac{1}{\mathbb{E}[T_j]} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)}.$$

Stationary distribution can be interpreted as the **long run proportion of time that the Markov chain is in state j.**

Heuristic proof

Step 1: Connecting long run proportion of time to inverse expected return time, i.e., we aim to show that for any state j, we have

$$\mathbb{P}_j \left[\lim_{n\to\infty} \frac{\sum_{i=1}^n 1\{X_i = j\}}{n} = \frac{1}{\mathbb{E}_j[T_j]} \right] = 1$$

If j is transient, both are 0.
If j is recurrent, see the next slide

When j is recurrent

Consider a Markov chain started from state j. Let S_k be the time till the k-th visit to state j. Then

$$S_k = T_{jj}(0) + T_{jj}(1) + \ldots + T_{jj}(k-1)$$

Here

- $T_{jj}(m) =$ the time between the mth and $(m+1)$st visit to state j.

Observe that $T_{jj}(0)$, $T_{jj}(1), \ldots T_{jj}(k-1)$ are i.i.d. and have the same distribution as T_i.
For k large, the Strong Law of Large Numbers tells us

$$\frac{1}{k}[T_{jj}(0) + T_{jj}(1) + \cdots + T_{jj}(k-1)] \to \mathbb{E}_j(T_j) \quad \text{almost surely}$$

i.e., the chain visits state j about k times in $k\mathbb{E}(T_j)$ steps.

Heuristic proof

Step 2: Connecting long run proportion of time to stationary probability
Consider a Markov chain starting from the stationary distribution.
Then in n steps, we expect about $n\pi(j)$ visits to the state j.
Hence

$$\pi_j$$

is roughly the proportion of time we see j.

Finite-State Markov Chains Have No Null Recurrent States

In an irreducible finite-state Markov chain all states are positive recurrent.

Proof.

Recall an irreducible Markov chain must be recurrent. (Why?)
Also recall that positive/null recurrence is a class property. Thus if one state is null recurrent, then all states are null recurrent.
In this case, we have for all j,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} 1\{X_i = j\}}{n} = 0.$$

Summing over j, we see the contradiction.

The Fundamental Limit Theorem of Markov Chain I

Let $\{X_n\}$ be an irreducible, positive recurrent, and aperiodic Markov chain. Then

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j, \qquad \text{for all } i, j.$$

Remark. For a finite state Markov chain, a limiting distribution exists if it is irreducible and aperiodic

Example 1: One-Dimensional Random Walk

In Lecture 4, we have shown that 1-dim symmetric random walk has no stationary distribution.

- Conclusion from 2nd limit theorem: 1-dim symmetric random walk is null recurrent, i.e.

$$\mathbb{E}[T_i] = \infty \quad \text{for all state } i$$

In fact, in Lecture 3 we have shown that

$$P_{ii}^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2}\left(\frac{1}{2}\right)^n \approx \sqrt{\frac{2}{\pi n}} & \text{if } n \text{ is even} \end{cases}$$

Thus we see $\lim_{n\to\infty} P_{ii}^{(n)} = 1/\mathbb{E}[T_i]$.

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$P_{i,i+1} = p \quad \text{for all } i = 0, 1, 2, \ldots$$
$$P_{i,i-1} = 1 - p \quad \text{for all } i = 1, 2, \ldots$$
$$p_{00} = 1 - p$$

Try to solve $\pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$

$$\begin{aligned}
\pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} = (1-p)(\pi_0 + \pi_1) &&\Rightarrow \pi_1 = \tfrac{p}{1-p}\pi_0 \\
\pi_1 &= \pi_0 P_{01} + \pi_2 P_{21} = p\pi_0 + (1-p)\pi_2 &&\Rightarrow \pi_2 = \left(\tfrac{p}{1-p}\right)^2 \pi_0 \\
\pi_2 &= \pi_0 P_{12} + \pi_3 P_{32} = p\pi_1 + (1-p)\pi_3 &&\Rightarrow \pi_3 = \left(\tfrac{p}{1-p}\right)^3 \pi_0 \\
&\vdots \\
\pi_j &= p\pi_{j-1} + (1-p)\pi_{j+1} &&\Rightarrow \pi_{j+1} = \left(\tfrac{p}{1-p}\right)^{j+1} \pi_0
\end{aligned}$$

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \left(\frac{p}{1-p} \right)^i = \begin{cases} \pi_0 \left(\frac{1-p}{1-2p} \right) & \text{if } p < 1/2 \\ \infty & \text{if } p \geq 1/2 \end{cases}$$

Conclusion: The process is positive recurrent iff $p < 1/2$, in which case

$$\pi_i = \frac{1-2p}{1-p} \left(\frac{p}{1-p} \right)^i, \quad i = 0, 1, 2, \ldots$$

Ex 3: Ehrenfest Diffusion Model with N Balls

Recall that in Lecture 4, we show that Ehrenfest Diffusion Model is irreducible, has period $= 2$, and there exists a solution to the set of equations

$$\pi_i \geq 0, \quad \sum_{i \in \mathfrak{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$$

which is

$$\pi_i = \binom{N}{i} \left(\frac{1}{2}\right)^N \quad \text{for } i = 0, 1, 2, \ldots, N$$

Though the limiting distribution $\lim_{n\to\infty} P_{ij}^{(n)}$ does not exist, we can show that

$$\lim_{n\to\infty} P_{ij}^{(2n)} = 2\binom{N}{j}(\frac{1}{2})^N, \quad \lim_{n\to\infty} P_{ij}^{(2n+1)} = 0 \quad \text{if } i+j \text{ is even}$$

$$\lim_{n\to\infty} P_{ij}^{(2n)} = 0, \quad \lim_{n\to\infty} P_{ij}^{(2n+1)} = 2\binom{N}{j}(\frac{1}{2})^N \quad \text{if } i+j \text{ is odd}$$

From the above, one can verify that
$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} = \binom{N}{j}(\frac{1}{2})^N = \pi_j$.

Exercise 4.50 on p.284

A Markov chain has transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

Communicating classes:

$$\begin{array}{ccc} \{1,2\} & \{3,4\} & \{5,6\} \\ \uparrow & \uparrow & \uparrow \\ \text{transient} & \text{recurrent} & \text{recurrent} \end{array}$$

Find $\lim_{n\to\infty} P^{(n)}$.

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is transient, hence,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is NOT accessible from i

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

The two classes {3,4} and {5,6} do not communicate and hence the transition probabilities in between are all 0.

Exercise 4.50 on p.284 (Cont'd)

Recall we have shown that the limiting distribution of a two-state Markov chain with the transition matrix $\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ is $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$. As the Markov chain restricted to the class {3,4} is also a Markov chain with the transition matrix

$$\begin{array}{c} \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \end{array}.$$

Hence,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

For the same reason,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{array}$$