STAT253/317 Lecture 6

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First step analysis

Utilize Recursive Relations of Markov Chains

Law of total expectation/variance

In many cases, we can use recursive relation to find $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$ without knowing the exact distribution of X_n.

$$\begin{aligned} \mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n]) \end{aligned}$$

Example 1: Simple Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } q = 1 - p \end{cases}$$

So

$$\begin{aligned} \mathbb{E}[X_{n+1}|X_n] &= p(X_n + 1) + q(X_n - 1) = X_n + p - q \\ \mathrm{Var}[X_{n+1}|X_n] &= 4pq \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = \mathbb{E}[X_n] + p - q \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n]) \\ &= \mathbb{E}[4pq] + \mathrm{Var}(X_n + p - q) = 4pq + \mathrm{Var}(X_n) \end{aligned}$$

So

$$\mathbb{E}[X_n] = n(p - q) + \mathbb{E}[X_0], \qquad \mathrm{Var}(X_n) = 4npq + \mathrm{Var}(X_0)$$

Example 2: Ehrenfest Urn Model with M Balls

Recall that

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{M-X_n}{M} \\ X_n - 1 & \text{with probability } \frac{X_n}{M} \end{cases}$$

We have

$$\mathbb{E}[X_{n+1}|X_n] = (X_n+1)\times\frac{M-X_n}{M}+(X_n-1)\times\frac{X_n}{M} = 1+\left(1-\frac{2}{M}\right)X_n.$$

Thus

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = 1 + \left(1-\frac{2}{M}\right)\mathbb{E}[X_n]$$

Subtracting $M/2$ from both sided of the equation above, we get

$$\mathbb{E}[X_{n+1}] - \frac{M}{2} = \left(1-\frac{2}{M}\right)(\mathbb{E}[X_n] - \frac{M}{2})$$

Thus

$$\mathbb{E}[X_n] - \frac{M}{2} = \left(1-\frac{2}{M}\right)^n(\mathbb{E}[X_0] - \frac{M}{2})$$

4.5.1 The Gambler's Ruin Problem

- A gambler repeatedly plays a game until he goes bankrupt or his fortune reaches N.
- In each game, he can win \$1 with probability p or lose \$1 with probability $q = 1 - p$.
- Outcomes of different games are independent
- Define $X_n =$ the gambler's fortune after the nth game.
- $\{X_n\}$ is a simple random walk w/ absorbing boundaries at 0 and N.

$$P_{00} = P_{NN} = 1,\ P_{i,i+1} = p, P_{i,i-1} = q,\ i = 1, 2, \ldots, N - 1$$

- Two recurrent classes: $\{0\}$ and $\{N\}$
  one transient class $\{1, 2, \ldots, N - 1\}$
- Regardless of the initial fortune X_0, eventually $\lim_{n\to\infty} X_n = 0$ or N as all states are transient except 0 or N.

4.5.1 The Gambler's Ruin Problem

Denote A as the event that the gambler's fortune reaches N before reaching 0. Then

$$P_i = P(A|X_0 = i).$$

Conditioning on the outcome of the first game,

$$\begin{aligned} P_i &= P(A|X_0 = i, \text{he wins the 1st game}) \underbrace{P(\text{he wins the 1st game})}_{=p} \\ &\quad + P(A|X_0 = i, \text{he loses the 1st game}) \underbrace{P(\text{he loses the 1st game})}_{=q} \\ &= P(A|X_0 = i, X_1 = i+1)p + P(A|X_0 = i, X_1 = i-1)q \\ &= \underbrace{P(A|X_1 = i+1)}_{=P_{i+1}} p + \underbrace{P(A|X_1 = i-1)}_{=P_{i-1}} q \ (\because \text{ Markov}) \end{aligned}$$

We get a set of equations

$$\begin{aligned} &P_i = pP_{i+1} + qP_{i-1} \quad \text{for } i = 1, 2, \ldots, N-1. \\ &P_0 = 0, \quad P_N = 1 \end{aligned}$$

Solving the equations $P_i = pP_{i+1} + qP_{i-1}$

$$
\begin{aligned}
(p+q)P_i &= pP_{i+1} + qP_{i-1} && \text{since } p+q=1 \\
\Leftrightarrow \quad q(P_i - P_{i-1}) &= p(P_{i+1} - P_i) \\
\Leftrightarrow \quad P_{i+1} - P_i &= (q/p)(P_i - P_{i-1})
\end{aligned}
$$

As $P_0 = 0$,

$$
\begin{aligned}
P_2 - P_1 &= (q/p)(P_1 - P_0) = (q/p)P_1 \\
P_3 - P_2 &= (q/p)(P_2 - P_1) = (q/p)^2 P_1 \\
&\vdots \\
P_i - P_{i-1} &= (q/p)(P_{i-1} - P_{i-2}) = (q/p)(q/p)^{i-2} P_1 = (q/p)^{i-1} P_1
\end{aligned}
$$

Adding up the equations above we get

$$P_i - P_1 = \left[q/p + (q/p)^2 + \cdots + (q/p)^{i-1}\right] P_1$$

From

$$P_i - P_1 = \left[q/p + (q/p)^2 + \cdots + (q/p)^{i-1}\right] P_1$$

we get

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1 & \text{if } p \neq q \\ i P_1 & \text{if } p = q \end{cases}$$

As $P_N = 1$, we get

$$P_1 = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N} & \text{if } p \neq 0.5 \\ 1/N & \text{if } p = 0.5 \end{cases}$$

So

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 0.5 \\ i/N & \text{if } p = 0.5 \end{cases}$$

If the gambler will never quit with whatever fortune he has ($N = \infty$), then

$$\lim_{N \to \infty} P_i = \begin{cases} 1 - (q/p)^i & \text{if } p > 0.5 \\ 0 & \text{if } p \leq 0.5 \end{cases}$$

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space $= \{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p,\ P_{i,i-1} = 1 - p = q,$ for $i = 1, 2, 3 \ldots$
- Only one class, irreducible
- For $i < j$, define

$$\begin{aligned} N_{ij} &= \min\{m > 0 : X_m = j | X_0 = i\} \\ &= \text{first time to reach state } j \text{ when starting from state } i \end{aligned}$$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$
  By the Markov property, N_{01}, $N_{12}, \ldots, N_{n-1,n}$ are indep.
- Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i + 1 \\ 1 + N^*_{i-1,i} + N^*_{i,i+1} & \text{if } X_1 = i - 1 \end{cases} \quad (1)$$

  Observe that $N^*_{i,i+1} \sim N_{i,i+1}$, and $N^*_{i,i+1}$ is indep of $N^*_{i-1,i}$.

4.5.3 Random Walk w/ Reflective Boundary at 0 (Cont'd)

Let $m_i = \mathbb{E}(N_{i,i+1})$. Taking expected value on Equation (1), we get

$$m_i = \mathbb{E}[N_{i,i+1}] = 1 + q\mathbb{E}[N^*_{i-1,i}] + q\mathbb{E}[N^*_{i,i+1}] = 1 + q(m_{i-1} + m_i)$$

Rearrange terms we get $pm_i = 1 + qm_{i-1}$ or

$$\begin{aligned} m_i &= \frac{1}{p} + \frac{q}{p}m_{i-1} \\ &= \frac{1}{p} + \frac{q}{p}(\frac{1}{p} + \frac{q}{p}m_{i-2}) \\ &= \frac{1}{p}\left[1 + \frac{q}{p} + (\frac{q}{p})^2 + \ldots + (\frac{q}{p})^{i-1}\right] + (\frac{q}{p})^i m_0 \end{aligned}$$

Since $N_{01} = 1$, which implies $m_0 = 1$.

$$m_i = \begin{cases} \frac{1-(q/p)^i}{p-q} + (\frac{q}{p})^i & \text{if } p \neq 0.5 \\ 2i + 1 & \text{if } p = 0.5 \end{cases}$$

Mean of $N_{0,n}$

Recall that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$

$$\begin{aligned}\mathbb{E}[N_{0n}] &= m_0 + m_1 + \ldots + m_{n-1} \\ &= \begin{cases} \frac{n}{p-q} - \frac{2pq}{(p-q)^2}[1 - (\frac{q}{p})^n] & \text{if } p \neq 0.5 \\ n^2 & \text{if } p = 0.5 \end{cases}\end{aligned}$$

When

$p > 0.5$	$\mathbb{E}[N_{0n}] \approx \frac{n}{p-q} - \frac{2pq}{(p-q)^2}$	linear in n
$p = 0.5$	$\mathbb{E}[N_{0n}] = n^2$	quadratic in n
$p < 0.5$	$\mathbb{E}[N_{0n}] = O(\frac{2pq}{(p-q)^2}(\frac{q}{p})^n)$	exponential in n

Example 3: Branching Processes (Section 4.7)

Consider a population of individuals.

- All individuals have the same lifetime
- Each individual will produce a random number of offsprings at the end of its life

Let $X_n =$ size of the n-th generation, $n = 0, 1, 2, \ldots$.
If $X_{n-1} = k$, the k individuals in the $(n-1)$-th generation will independently produce $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,k}$ new offsprings, and $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,X_{n-1}}$ are i.i.d such that

$$P(Z_{n,i} = j) = P_j, \ j \geq 0.$$

We suppose that $P_j < 1$ for all $j \geq 0$.

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i} \tag{2}$$

$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \ldots\}$.

Mean of a Branching Process

Let $\mu = \mathbb{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. Since $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathbb{E}[X_n|X_{n-1}] = \mathbb{E}\left[\sum_{i=1}^{X_{n-1}} Z_{n,i}\middle|X_{n-1}\right] = X_{n-1}\mathbb{E}[Z_{n,i}] = X_{n-1}\mu$$

So

$$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n|X_{n-1}]] = \mathbb{E}[X_{n-1}\mu] = \mu\mathbb{E}[X_{n-1}]$$

If $X_0 = 1$, then

$$\mathbb{E}[X_n] = \mu\mathbb{E}[X_{n-1}] = \mu^2\mathbb{E}[X_{n-2}] = \ldots = \mu^n\mathbb{E}[X_0]$$

- If $\mu < 1 \Rightarrow \mathbb{E}[X_n] \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} \mathrm{P}(X_n \geq 1) = 0$ the branching processes will eventually die out.
- What if $\mu = 1$ or $\mu > 1$?

Variance of a Branching Process

Let $\sigma^2 = \mathrm{Var}[Z_{n,i}] = \sum_{j=0}^{\infty}(j-\mu)^2 P_j$. $\mathrm{Var}(X_n)$ may be obtained using the conditional variance formula

$$\mathrm{Var}(X_n) = \mathbb{E}[\mathrm{Var}(X_n|X_{n-1})] + \mathrm{Var}(\mathbb{E}[X_n|X_{n-1}]).$$

Again from that $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathbb{E}[X_n|X_{n-1}] = X_{n-1}\mu, \quad \mathrm{Var}(X_n|X_{n-1}) = X_{n-1}\sigma^2$$

and hence

$$\begin{aligned}\mathrm{Var}(\mathbb{E}[X_n|X_{n-1}]) &= \mathrm{Var}(X_{n-1}\mu) = \mu^2\mathrm{Var}(X_{n-1})\\ \mathbb{E}[\mathrm{Var}(X_n|X_{n-1})] &= \sigma^2\mathbb{E}[X_{n-1}] = \sigma^2\mu^{n-1}\mathbb{E}[X_0].\end{aligned}$$

So

$$\begin{aligned}\mathrm{Var}(X_n) &= \sigma^2\mu^{n-1}\mathbb{E}[X_0] + \mu^2\mathrm{Var}(X_{n-1})\\ &= \sigma^2\mathbb{E}[X_0](\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}) + \mu^{2n}\mathrm{Var}(X_0)\\ &= \begin{cases}\sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right)\mathbb{E}[X_0] + \mu^{2n}\mathrm{Var}(X_0) & \text{if } \mu \neq 1\\ n\sigma^2\mathbb{E}[X_0] + \mu^{2n}\mathrm{Var}(X_0) & \text{if } \mu = 1\end{cases}\end{aligned}$$