STAT253/317 Lecture 7 Generating Functions

For a non-negative integer-valued random variable T, the generating function of T is the expected value of s^T as a function of s

$$G(s) = \mathsf{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k),$$

in which s^T is defined as 0 if $T = \infty$.
Since $0 \leq \mathrm{P}(T = k) \leq 1$, the generating function is always well-defined for $-1 \leq s \leq 1$

Examples of Generating Functions

- If T has a geometric distribution: $P(T = k) = p(1-p)^k$, $k = 0, 1, 2, \ldots$, the generating function of T is

$$G(s) = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k) = \sum_{k=0}^{\infty} s^k p(1-p)^k = \frac{p}{1-(1-p)s}$$

- If T has a Binomial distribution $P(T = k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2, \ldots, n$, the generating function of T is

$$G(s) = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k) = \sum_{k=0}^{\infty} s^k \binom{n}{k} p^k (1-p)^{n-k}$$
$$= (ps + (1-p))^n$$

Properties of Generating Function

$$G(s) = \mathsf{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k)$$

- $G(s)$ is a power series converging absolutely for all $-1 \leq s \leq 1$. since $0 \leq \mathrm{P}(T = k) \leq 1$ and $\sum_k \mathrm{P}(T = k) \leq 1$.
- $G(1) = \mathrm{P}(T < \infty) \begin{cases} = 1 & \text{if } T \text{ is finite w/ prob. 1} \\ < 1 & \text{otherwise} \end{cases}$
- $P(T = k) = \dfrac{G^{(k)}(0)}{k!}$

  Knowing $G(s) \Leftrightarrow$ Knowing $\mathrm{P}(T = k)$ for all $k = 0, 1, 2, \ldots$

More Properties of Generating Functions

$$G(s) = \mathsf{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k)$$

- $\mathsf{E}[T] = \lim_{s \to 1^-} G'(s)$ if it exists because

$$G'(s) = \frac{d}{ds}\mathsf{E}[s^T] = \mathsf{E}[Ts^{T-1}] = \sum_{k=1}^{\infty} s^{k-1} k \mathrm{P}(T = k).$$

- $\mathsf{E}[T(T-1)] = \lim_{s \to 1^-} G''(s)$ if it exists because

$$G''(s) = \mathsf{E}[T(T-1)s^{T-2}] = \sum\nolimits_{k=2}^{\infty} s^{k-2} k(k-1)\mathrm{P}(T = k)$$

- If T and U are **independent** non-negative-integer-valued random variables, with generating function $G_T(s)$ and $G_U(s)$ respectively, then the generating function of $T + U$ is

$$G_{T+U}(s) = \mathsf{E}[s^{T+U}] = \mathsf{E}[s^T]\mathsf{E}[s^U] = G_T(s)G_U(s)$$

Example 3: Branching Processes (Section 4.7)

Consider a population of individuals.

- All individuals have the same lifetime
- Each individual will produce a random number of offsprings at the end of its life

Let X_n = size of the n-th generation, $n = 0, 1, 2, \ldots$.
If $X_{n-1} = k$, the k individuals in the $(n-1)$-th generation will independently produce $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,k}$ new offsprings, and $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,X_{n-1}}$ are i.i.d such that

$$P(Z_{n,i} = j) = P_j, \; j \geq 0.$$

We suppose that $P_j < 1$ for all $j \geq 0$.

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i} \tag{1}$$

$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \ldots\}$.

Extinction Probability of a Branching Process

$$\begin{aligned}\text{Let } \pi_0 &= \lim_{n\to\infty} \mathrm{P}(X_n = 0 | X_0 = 1) \\ &= \mathrm{P}(\text{the population will eventually die out} | X_0 = 1)\end{aligned}$$

Mean of a Branching Process

Let $\mu = \mathsf{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. Since $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathsf{E}[X_n | X_{n-1}] = \mathsf{E}\left[\sum_{i=1}^{X_{n-1}} Z_{n,i} \middle| X_{n-1}\right] = X_{n-1}\mathsf{E}[Z_{n,i}] = X_{n-1}\mu$$

So

$$\mathsf{E}[X_n] = \mathsf{E}[\mathsf{E}[X_n | X_{n-1}]] = \mathsf{E}[X_{n-1}\mu] = \mu\mathsf{E}[X_{n-1}]$$

If $X_0 = 1$, then

$$\mathsf{E}[X_n] = \mu\mathsf{E}[X_{n-1}] = \mu^2\mathsf{E}[X_{n-2}] = \ldots = \mu^n$$

- If $\mu < 1 \Rightarrow \mathsf{E}[X_n] \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} \mathrm{P}(X_n \geq 1) = 0$ the branching processes will eventually die out.
- What if $\mu = 1$ or $\mu > 1$?

Variance of a Branching Process

Let $\sigma^2 = \mathsf{Var}[Z_{n,i}] = \sum_{j=0}^{\infty}(j-\mu)^2 P_j$. $\mathsf{Var}(X_n)$ may be obtained using the conditional variance formula

$$\mathsf{Var}(X_n) = \mathsf{E}[\mathsf{Var}(X_n|X_{n-1})] + \mathsf{Var}(\mathsf{E}[X_n|X_{n-1}]).$$

Again from that $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathsf{E}[X_n|X_{n-1}] = X_{n-1}\mu, \quad \mathsf{Var}(X_n|X_{n-1}) = X_{n-1}\sigma^2$$

and hence

$$\begin{aligned}\mathsf{Var}(\mathsf{E}[X_n|X_{n-1}]) &= \mathsf{Var}(X_{n-1}\mu) = \mu^2\mathsf{Var}(X_{n-1})\\ \mathsf{E}[\mathsf{Var}(X_n|X_{n-1})] &= \sigma^2\mathsf{E}[X_{n-1}] = \sigma^2\mu^{n-1}\mathsf{E}[X_0].\end{aligned}$$

So

$$\begin{aligned}\mathsf{Var}(X_n) &= \sigma^2\mu^{n-1}\mathsf{E}[X_0] + \mu^2\mathsf{Var}(X_{n-1})\\ &= \sigma^2\mathsf{E}[X_0](\mu^{n-1}+\mu^n+\ldots+\mu^{2n-2}) + \mu^{2n}\mathsf{Var}(X_0)\\ &= \begin{cases}\sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right)\mathsf{E}[X_0] + \mu^{2n}\mathsf{Var}(X_0) & \text{if } \mu \neq 1\\ n\sigma^2\mathsf{E}[X_0] + \mu^{2n}\mathsf{Var}(X_0) & \text{if } \mu = 1\end{cases}\end{aligned}$$

Generating Functions of the Branching Processes

Let $g(s) = \mathsf{E}[s^{Z_{n,i}}] = \sum_{k=0}^{\infty} P_k s^k$ be the generating function of $Z_{n,i}$, and $G_n(s)$ be the generating function of X_n, $n = 0, 1, 2, \ldots$. Then $\{G_n(s)\}$ satisfies the following two iterative equations.

(i) $G_{n+1}(s) = G_n(g(s))$ for $n = 0, 1, 2, \ldots$

(ii) $G_{n+1}(s) = g(G_n(s))$ if $X_0 = 1$, for $n = 0, 1, 2, \ldots$

Proof of (i).

$$\begin{aligned}
\mathsf{E}[s^{X_{n+1}}|X_n] = \mathsf{E}\left[s^{\sum_{i=1}^{X_n} Z_{n,i}}\right] &= \mathsf{E}\Big[\prod_{i=1}^{X_n} s^{Z_{n,i}}\Big] \\
&= \prod_{i=1}^{X_n} \mathsf{E}[s^{Z_{n,i}}] \qquad \text{by indep. of } Z_{n,i}\text{'s} \\
&= \prod_{i=1}^{X_n} g(s) \qquad \text{as } g(s) = \mathsf{E}[s^{Z_{n,i}}] \\
&= g(s)^{X_n}
\end{aligned}$$

From which, we have

$$G_{n+1}(s) = \mathsf{E}[s^{X_{n+1}}] = \mathsf{E}[\mathsf{E}[s^{X_{n+1}}|X_n]] = \mathsf{E}[g(s)^{X_n}] = G_n(g(s))$$

since $G_n(s) = \mathsf{E}[s^{X_n}]$.

Proof of (ii) $G_{n+1}(s) = g(G_n(s))$ if $X_0 = 1$

Suppose there are k individuals in the first generation ($X_1 = k$). Let Y_i be the number offspring of the ith individual in the first generation in the $(n+1)$st generation. Obviously,

$$X_{n+1} = Y_1 + \ldots + Y_k.$$

Observe $Y_1, \ldots, Y_k$'s are indep and each has the same distn. as X_n since they are all the size of the nth generation of a single ancestor. Thus, by indep. of Y_i's

$$\mathsf{E}[s^{X_{n+1}}|X_1 = k] = \mathsf{E}\left[s^{Y_1+\ldots+Y_k}\right] = \mathsf{E}\left[\prod_{i=1}^{k} s^{Y_i}\right] = \prod_{i=1}^{k} \mathsf{E}[s^{Y_i}]$$

Since Y_i's have the same dist'n as X_n and $G_n(s) = \mathsf{E}[s^{X_n}]$, we have

$$\mathsf{E}[s^{X_{n+1}}|X_1 = k] = \prod_{i=1}^{k} G_n(s) = (G_n(s))^k$$

Since $X_0 = 1$, $X_1 = Z_{1,1}$, and hence $\mathrm{P}(X_1 = k) = P_k$.

$$G_{n+1}(s) = \mathsf{E}[s^{X_{n+1}}] = \sum_{k=0}^{\infty} \mathsf{E}[s^{X_{n+1}}|X_1 = k]P_k = \sum_{k=0}^{\infty}(G_n(s))^k P_k = g(G_n(s)),$$

where the last equality comes from that $g(s) = \sum_{k=0}^{\infty} P_k s^k$.

Example: calculating distributions of X_n

Suppose $X_0 = 1$, and $(P_0, P_1, P_2) = (1/4, 1/2, 1/4)$. Find the distribution of X_2.

Sol.

$$g(s) = \frac{1}{4}s^0 + \frac{1}{2}s^1 + \frac{1}{4}s^2 = (1+s)^2/4.$$

Since $X_0 = 1$, $G_0(s) = \mathsf{E}[s^{X_0}] = \mathsf{E}[s^1] = s$. From (i) we have

$$\begin{aligned} G_1(s) &= G_0(g(s)) = g(s) = (1+s)^2/4 \\ G_2(s) &= G_1(g(s)) = \frac{1}{4}(1 + \frac{1}{4}(1+s)^2)^2 = \frac{1}{64}(5 + 2s + s^2)^2 \\ &= \frac{1}{64}(25 + 20s + 14s^2 + 4s^3 + s^4) = \sum_{k=0}^{\infty} \mathrm{P}(X_2 = k)s^k \end{aligned}$$

The coefficient of s^k in the polynomial of $G_2(s)$ is the chance that $X_2 = k$.

k	0	1	2	3	4
$\mathrm{P}(X_2 = k)$	$\frac{25}{64}$	$\frac{20}{64}$	$\frac{14}{64}$	$\frac{4}{64}$	$\frac{1}{64}$

and $\mathrm{P}(X_2 = k) = 0$ for $k \geq 5$.

Extinction Probability of a Branching Process

$$\begin{aligned} \text{Let } \pi_0 &= \lim_{n\to\infty} \mathrm{P}(X_n = 0 | X_0 = 1) \\ &= \mathrm{P}(\text{the population will eventually die out} | X_0 = 1) \end{aligned}$$

As $G_n(s) = \mathsf{E}[s^{X_n}] = \sum_{k=0}^{\infty} P(X_n = k)s^k$, plugging in $s = 0$, we get

$$G_n(0) = P(X_n = 0) = P(\text{extinct by the } n\text{th generation}).$$

Recall that if $X_0 = 1$, $G_1(s) = g(s)$, and $G_{n+1}(s) = g(G_n(s))$. We can compute $G_n(0)$ iteratively as follows

$$\begin{aligned} G_1(0) &= g(0) \\ G_{n+1}(0) &= g(G_n(0)), \quad n = 1, 2, 3, \ldots \end{aligned}$$

Finally, we can get the extinction probability by taking the limit

$$\pi_0 = \lim_{n\to\infty} G_n(0).$$

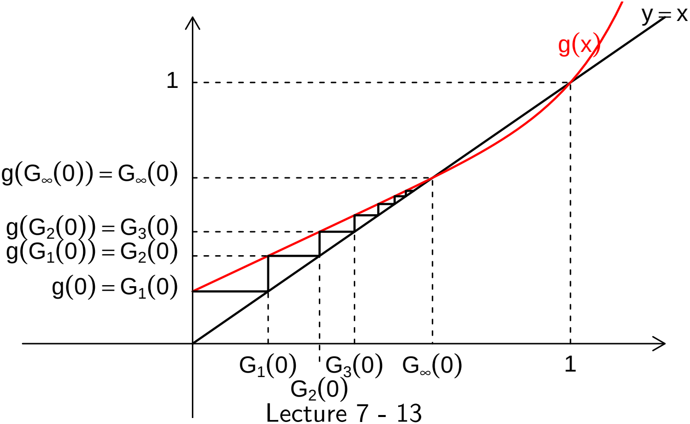
Extinction Probability of a Branching Process

If $X_0 = 1$, the extinction probability π_0 is a **smallest root** of the equation

$$g(s) = s \qquad (2)$$

in the range $0 < s \leq 1$, where $g(s) = \sum_{k=0}^{\infty} P_k s^k$ is the generating function of $Z_{n,i}$.

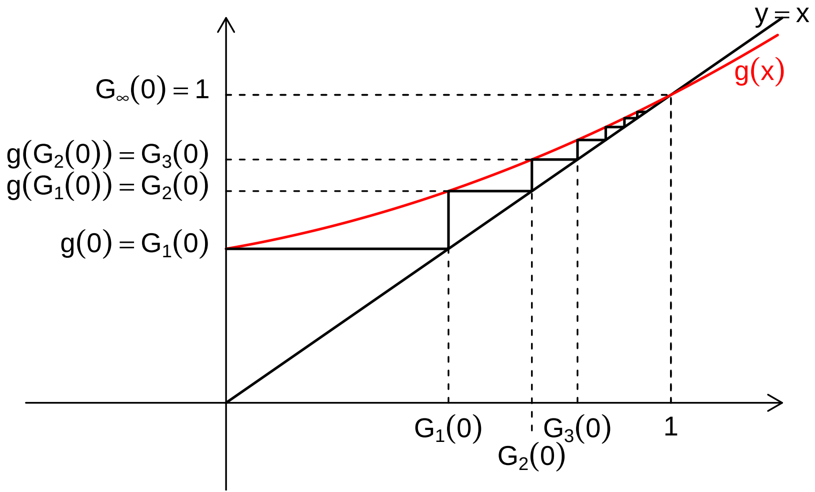
Proof.



A Branching Process Will Become Extinct If $\mu \leq 1$

Let $\mu = \mathsf{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. If $\mu \leq 1$, the extinction probability π_0 is 1.

Proof.



Formal Proof

Let $h(s) = g(s) - s$. Since $g(1) = 1$, $g'(1) = \mu$,

$$h(1) = g(1) - 1 = 0,$$

$$h'(s) = \left(\sum_{j=1}^{\infty} jP_j s^{j-1}\right) - 1 \leq \left(\sum_{j=1}^{\infty} jP_j\right) - 1 = \mu - 1 \quad \text{for } 0 \leq s < 1$$

Thus $\mu \leq 1 \Rightarrow h'(s) \leq 0$ for $0 \leq s < 1$

$\Rightarrow h(s)$ is non-increasing in $[0, 1)$

$\Rightarrow h(s) > h(1) = 0$ for $0 \leq s < 1$

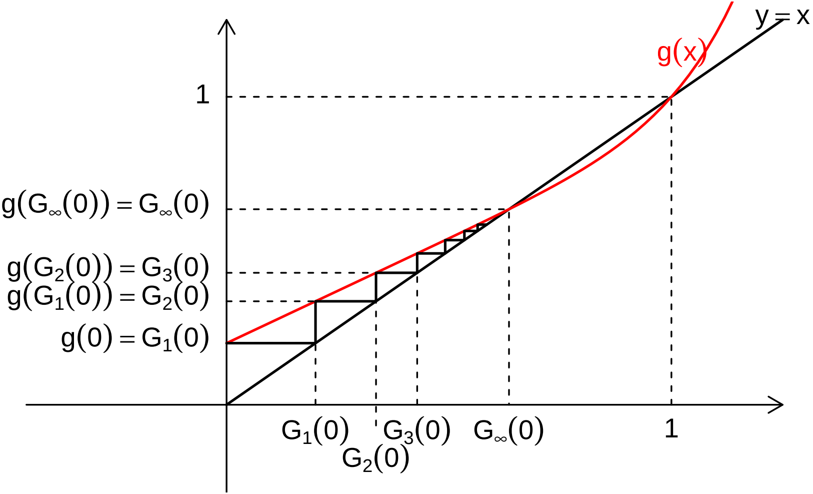
$\Rightarrow g(s) > s$ for $0 \leq s < 1$

$\Rightarrow$ There is no root in [0,1).

Extinction Probability When $\mu > 1$

If $\mu > 1$, there is a unique root of the equation $g(s) = s$ in the domain $[0, 1)$, and that is the extinction probability.

Proof.



Formal Proof

Let $h(s) = g(s) - s$. Observe that

$$\begin{aligned} h(0) &= g(0) = P_0 > 0 \\ h'(0) &= g'(0) - 1 = P_1 - 1 < 0 \\ \text{Then } \mu > 1 &\Rightarrow h'(1) = \mu - 1 > 0 \\ &\Rightarrow h(s) \text{ is increasing near 1} \\ &\Rightarrow h(1-\delta) < h(1) = 0 \text{ for } \delta > 0 \text{ small enough} \end{aligned}$$

Since $h(s)$ is continuous in $[0, 1)$, there must be a root to $h(s) = s$. The root is unique since

$$h''(s) = g''(s) = \sum_{j=2}^{\infty} j(j-1)P_j s^{j-2} \geq 0 \quad \text{for } 0 \leq s < 1$$

$h(s)$ is convex in [0,1).

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space $= \{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p,\ P_{i,i-1} = 1 - p = q,$ for $i = 1, 2, 3 \ldots$
- Only one class, irreducible
- For $i < j$, define

$$\begin{aligned} N_{ij} &= \min\{m > 0 : X_m = j | X_0 = i\} \\ &= \text{time to reach state } j \text{ starting in state } i \end{aligned}$$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$
  By the Markov property, N_{01}, $N_{12}, \ldots, N_{n-1,n}$ are indep.
- Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i + 1 \\ 1 + N^*_{i-1,i} + N^*_{i,i+1} & \text{if } X_1 = i - 1 \end{cases} \tag{3}$$

where $N^*_{i-1,i} \sim N_{i-1,i}$, $N^*_{i,i+1} \sim N_{i,i+1}$, and $N^*_{i-1,i}$, $N^*_{i,i+1}$ are indep.

Generating Function of $N_{i,i+1}$

Let $G_i(s)$ be the generating function of $N_{i,i+1}$. From (3), and by the independence of $N^*_{i-1,i}$ and $N^*_{i,i+1}$, we get that

$$G_i(s) = ps + q\mathsf{E}[s^{1+N^*_{i-1,i}+N^*_{i,i+1}}] = ps + qsG_{i-1}(s)G_i(s)$$

So

$$G_i(s) = \frac{ps}{1 - qsG_{i-1}(s)} \tag{4}$$

Since N_{01} is always 1, we have $G_0(s) = s$. Using the iterative relation (4), we can find

$$G_1(s) = \frac{ps}{1 - qsG_0(s)} = \frac{ps}{1 - qs^2} = ps\sum_{k=0}^{\infty}(qs^2)^k = \sum_{k=0}^{\infty} pq^k s^{2k+1}$$

So $\mathrm{P}(N_{12} = n) = \begin{cases} pq^k & \text{if } n = 2k+1 \text{ for } k = 0, 1, 2 \ldots \\ 0 & \text{if } n \text{ is even} \end{cases}$

Similarly,

$$\begin{aligned}
G_2(s) &= \frac{ps}{1 - qsG_1(s)} = \frac{ps(1 - qs^2)}{1 - q(1+p)s^2} \\
&= \frac{ps}{1 - q(1+p)s^2} - \frac{pqs^3}{1 - q(1+p)s^2} \\
&= ps \sum_{k=0}^{\infty} (q(1+p)s^2)^k - pqs^3 \sum_{k=0}^{\infty} (q(1+p)s^2)^k \\
&= \sum_{k=0}^{\infty} pq^k (1+p)^k s^{2k+1} - \sum_{k=0}^{\infty} pq^{k+1} (1+p)^k s^{2k+3} \\
&= ps + \sum_{k=1}^{\infty} pq^k [(1+p)^k - (1+p)^{k-1}] s^{2k+1} \\
&= ps + \sum_{k=1}^{\infty} p^2 q^k (1+p)^{k-1} s^{2k+1}
\end{aligned}$$

So

$$\mathrm{P}(N_{23} = n) = \begin{cases} p & \text{if } n = 1 \\ p^2 q^k (1+p)^{k-1} & \text{if } n = 2k+1 \text{ for } k = 1, 2, \dots \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Mean of $N_{i,i+1}$

Recall that $G_i'(1) = E(N_{i,i+1})$. Let $m_i = E(N_{i,i+1}) = G_i'(1)$.

$$G_i'(s) = \frac{p(1 - qsG_{i-1}(s)) + ps(qG_{i-1}(s) + qsG_{i-1}'(s))}{(1 - qsG_{i-1}(s))^2}$$

$$= \frac{p + pqs^2G_{i-1}'(s)}{(1 - qsG_{i-1}(s))^2}$$

Since $N_{i,i+1} < \infty$, $G_i(1) = 1$ for all $i = 0, 1, \ldots, n-1$. We have

$$m_i = G_i'(1) = \frac{p + pqG_{i-1}'(1)}{(1-q)^2} = \frac{1 + qG_{i-1}'(1)}{p} = \frac{1}{p} + \frac{q}{p}m_{i-1}$$

We get the same iterative equation as in Lecture 7.