STAT253/317 Lecture 8

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Chapter 5 Poisson Processes

5.2 Exponential Distribution

Let X follow exponential distribution with rate λ: $X \sim Exp(\lambda)$.

- Density: $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- CDF: $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$
- $\mathbb{E}(X) = 1/\lambda$, $\mathrm{V}ar(X) = 1/\lambda^2$
- If $X_1, \ldots, X_n$ are i.i.d $Exp(\lambda)$, then $S_n = X_1 + \cdots + X_n \sim Gamma(n, \lambda)$, with density

$$f_{S_n}(x) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

The Exponential Distribution is Memoryless ($\star\star\star\star\star$)

Lemma: for all $s, t \geq 0$

$$\mathrm{P}(X > t + s \mid X > t) = \mathrm{P}(X > s)$$

Proof.

$$\begin{aligned}
\mathrm{P}(X > t + s | X > t) &= \frac{\mathrm{P}(X > t + s \text{ and } X > t)}{\mathrm{P}(X > t)} \\
&= \frac{\mathrm{P}(X > t + s)}{\mathrm{P}(X > t)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathrm{P}(X > s)
\end{aligned}$$

Implication. If the lifetime of batteries has an Exponential distribution, then *a used battery is as good as a new one*, as long as it's not dead!

Another Important Property of the Exponential

If $X_1, \dots, X_n$ are independent, $X_i, \sim Exp(\lambda_i)$ for $i = 1, \dots, n$ then

(i) $\min(X_1, \dots, X_n) \sim Exp(\lambda_1 + \dots + \lambda_n)$, and

(ii) $\mathrm{P}\big(X_j = \min(X_1, \dots, X_n)\big) = \dfrac{\lambda_j}{\lambda_1 + \dots + \lambda_n}$

Proof of (i)

$$\begin{aligned} &\mathrm{P}(\min(X_1, \dots, X_n) > t) = \mathrm{P}(X_1 > t, \dots, X_n > t) \\ &= \mathrm{P}(X_1 > t) \dots \mathrm{P}(X_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} \\ &= e^{-(\lambda_1 + \dots + \lambda_n)t}. \end{aligned}$$

Proof of (ii)

$$
\begin{aligned}
& \mathrm{P}\big(X_j = \min(X_1, \ldots, X_n)\big) \\
&= \mathrm{P}(X_j < X_i \text{ for } i = 1, \ldots, n, i \neq j) \\
&= \int_0^\infty \mathrm{P}(X_j < X_i \text{ for } i \neq j | X_j = t) \lambda_j e^{-\lambda_j t} dt \\
&= \int_0^\infty \mathrm{P}(t < X_i \text{ for } i \neq j) \lambda_j e^{-\lambda_j t} dt \\
&= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} \mathrm{P}(X_i > t) dt \\
&= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} e^{-\lambda_i t} dt \\
&= \lambda_j \int_0^\infty e^{-(\lambda_1 + \cdots + \lambda_n) t} dt \\
&= \frac{\lambda_j}{\lambda_1 + \cdots + \lambda_n}
\end{aligned}
$$

Example 5.8: Post Office

- A post office has two clerks.
- Service times for clerk $i \sim Exp(\lambda_i)$, $i = 1$, 2
- When you arrive, both clerks are busy but no one else waiting. You will enter service when either clerk becomes free.
- Find $\mathbb{E}[T]$, where $T =$ the amount of time you spend in the post office.

Solution. Let $R_i =$ remaining service time of the customer with clerk i, $i = 1$, 2.

- Note R_i's are indep. $\sim Exp(\lambda_i)$, $i = 1$, 2 by the memoryless property
- Observe $T = \min(R_1, R_2) + S$ where S is your service time
- Using the property of exponential distributions,

$$\min(R_1, R_2) \sim Exp(\lambda_1 + \lambda_2) \quad \Rightarrow \quad \mathbb{E}[\min(R_1, R_2)] = \frac{1}{\lambda_1 + \lambda_2}$$

Example 5.8: Post Office (Cont'd)

As for your service time S, observe that

$$S \sim \begin{cases} Exp(\lambda_1) & \text{if } R_1 < R_2 \\ Exp(\lambda_2) & \text{if } R_2 < R_1 \end{cases} \Rightarrow \begin{array}{l} \mathbb{E}[S|R_1 < R_2] = 1/\lambda_1 \\ \mathbb{E}[S|R_2 < R_1] = 1/\lambda_2 \end{array}$$

Recall that $\mathrm{P}(R_1 < R_2) = \lambda_1/(\lambda_1 + \lambda_2)$ So

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[S|R_1 < R_2]\mathrm{P}(R_1 < R_2) + \mathbb{E}[S|R_2 < R_1]\mathrm{P}(R_2 < R_1) \\ &= \frac{1}{\lambda_1} \times \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2} \times \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{2}{\lambda_1 + \lambda_2} \end{aligned}$$

Hence the expected amount of time you spend in the post office is

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}[\min(R_1, R_2)] + \mathbb{E}[S] \\ &= \frac{1}{\lambda_1 + \lambda_2} + \frac{2}{\lambda_1 + \lambda_2} = \frac{3}{\lambda_1 + \lambda_2}. \end{aligned}$$

5.3.1. Counting Processes

A counting process $\{N(t)\}$ is a cumulative count of number of events happened up to time t.

Definition.
A stochastic processes $\{N(t), t \geq 0\}$ is a *counting process* satisfying

(i) $N(t) = 0, 1, \ldots$ (integer valued),

(ii) If $s < t$, then $N(s) \leq N(t)$.

(iii) For $s < t$, $N(t) - N(s) =$ number of events that occur in the interval $(s, t]$.

Definition.

A process $\{X(t), t \geq 0\}$ is said to have *stationary increments* if for any $t > s$, the distribution of $X(t) - X(s)$ depends on s and t only through the difference $t - s$, for all $s < t$.
That is, $X(t + a) - X(s + a)$ has the same distribution as $X(t) - X(s)$ for any constant a.

Definition.

A process $\{X(t), t \geq 0\}$ is said to have *independent increments* if for any $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k$, the random variable $X(t_1) - X(s_1)$, $X(t_2) - X(s_2), \ldots, X(t_k) - X(s_k)$ are independent, i.e. the numbers of events that occur in **disjoint** time intervals are **independent**.

Example. Modified simple random walk $\{X_n, n \geq 0\}$ is a process with independent and stationary increment, since $X_n = \sum_{k=0}^{n} \xi_k$ where ξ_k's are i.i.d with $\mathrm{P}(\xi_k = 1) = p$ and $\mathrm{P}(\xi_k = 0) = 1 - p$.

Definition 5.1 of Poisson Processes

A Poisson process with rate $\lambda > 0$ $\{N(t), t \geq 0\}$ is a counting process satisfying

(i) $N(0) = 0$,

(ii) For $s < t$, $N(t) - N(s)$ is independent of $N(s)$ (independent increment)

(iii) For $s < t$, $N(t) - N(s) \sim Poi(\lambda(t-s))$, i.e.,

$$\mathrm{P}(N(t) - N(s) = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$$

Remark: In (iii), the distribution of $N(t) - N(s)$ depends on $t - s$ only, not s, which implies $N(t)$ has stationary increment.

Definition 5.3 of Poisson Processes

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) The process has stationary and independent increments.

(iii) $\mathrm{P}(N(h) = 1) = \lambda h + o(h)$.

(iv) $\mathrm{P}(N(h) \geq 2) = o(h)$.

Theorem 5.1 Definitions 5.1 and 5.3 are equivalent.
[Proof of Definitions 5.1 $\Rightarrow$ Definition 5.3]
From Definitions 5.1, $N(h) \sim Poi(h)$. Thus

$$\begin{aligned} \mathrm{P}(N(h) = 1) &= \lambda h e^{-\lambda h} = \lambda h + o(h) \\ \mathrm{P}(N(h) \geq 2) &= 1 - \mathrm{P}(N(h) = 0) - \mathrm{P}(N(h) = 1) \\ &= 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} = o(h) \end{aligned}$$

Proof of Definitions 5.3 $\Rightarrow$ Definition 5.1:
See textbook.

Arrival & Interarrival Times of Poisson Processes

Let

$$\begin{aligned} S_n &= \text{Arrival time of the } n\text{-th event}, n = 1, 2, \ldots \\ T_1 &= S_1 = \text{Time until the 1st event occurs} \\ T_n &= S_n - S_{n-1} \\ &= \text{time elapsed between the } (n-1)\text{st and } n\text{-th event}, \\ &\quad n = 2, 3, \ldots \end{aligned}$$

Proposition 5.1

The interarrival times $T_1, T_2, \ldots, T_k, \ldots,$ are i.i.d $\sim Exp(\lambda)$.

Consequently, as the distribution of the sum of n i.i.d Exp(λ) is $Gamma(n, \lambda)$, the arrival time of the nth event is

$$S_n = \sum_{i=1}^{n} T_i \sim Gamma(n, \lambda)$$

Proof of Proposition 5.1

$$
\begin{aligned}
& \mathrm{P}(T_{n+1} > t | T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
= & \mathrm{P}(0 \text{ event in } (s_n, s_n + t] | T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
& \qquad\qquad\qquad\qquad (\text{where } s_n = t_1 + t_2 + \cdots + t_n) \\
= & \mathrm{P}(0 \text{ event in } (s_n, s_n + t]) \quad (\text{by indep increment}) \\
= & \mathrm{P}(N(s_n + t) - N(s_n) = 0) \\
= & e^{-\lambda t}
\end{aligned}
$$

where the last step comes from the fact that

- $N(s_n + t) - N(s_n) \sim$ Poisson(λt) and
- $P(N = k) = e^{-\mu}\mu^k/k!$ if $N \sim$ Poisson(μ), $k = 0, 1, 2 \ldots$

This shows that T_{n+1} is $\sim Exp(\lambda)$, and is independent of $T_1, T_2, \ldots, T_n$.

Definition 3 of the Poisson Process

A continuous-time stochastic process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

(i) $N(0) = 0$,

(ii) $N(t)$ counts the number of events that have occurred up to time t (i.e., it is a counting process).

(iii) The times between events are independent and identically distributed with an Exp(λ) distribution.

We have seen how Definition 5.1 implies (i), (ii) and (iii) in Definition 3. The proof of the converse is omitted.