STAT253/317 Lecture 9

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5.3 The Poisson Processes

Properties of Poisson Processes

Outline:

- Conditional Distribution of the Arrival Times
- Superposition & Thinning
- "Converse" of Superposition & Thinning

5.3.5 Conditional Distribution of Arrival Times is Uniform

Given $N(t) = 1$, then T_1, the arrival time of the first event $\sim$ Uniform$(0, t)$

Proof. For $s < t$,

$$\mathrm{P}(T_1 \leq s | N(t) = 1) = \frac{\mathrm{P}(T_1 \leq s, N(t) = 1)}{\mathrm{P}(N(t) = 1)}$$

$$= \frac{\mathrm{P}(\text{1 event in } (0, s], \text{ no events in } (s, t])}{\mathrm{P}(N(t) = 1)}$$

$$= \frac{\mathrm{P}(N(s) = 1)\mathrm{P}(N(t) - N(s) = 0)}{\mathrm{P}(N(t) = 1)} \text{ by indep. increment}$$

$$=^* \frac{(\lambda s e^{-\lambda s})(e^{-\lambda(t-s)})}{\lambda t e^{-\lambda t}} = \frac{s}{t}, \quad s < t.$$

where the step $=^*$ comes from the fact that

- $N(s) \sim$ Poisson(λs), $N(t) - N(s) \sim$ Poisson$(\lambda(t - s))$, and $N(t) \sim$ Poisson(λt)
- $P(N = k) = e^{-\mu} \mu^k / k!$ if $N \sim$ Poisson(μ), $k = 0, 1, 2 \ldots$

Review of Order Statistics

Suppose X_1, X_2,..., X_n are i.i.d. random variables with a common density $f(x)$. Their joint density would be the product of the marginal density

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n).$$

Let $X_{(i)}$ be the ith smallest number among $X_1, X_2, \ldots, X_n$. $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ is called the order statistics of $X_1, X_2, \ldots, X_n$

- $X_{(1)}$ is the minimum
- $X_{(n)}$ is the maximum
- $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$

The joint density of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is

$$h(x_1, x_2, \ldots, x_n) = \begin{cases} n! f(x_1)f(x_2)\ldots f(x_n), & \text{if } x_1 \leq x_2 \leq \ldots \leq x_n. \\ 0 & \text{otherwise} \end{cases}$$

Example

If $U_1, U_2, \ldots, U_n$ are indep. Uniform$(0, t)$, their common density is

$$f(u) = \begin{cases} 1/t, & \text{for } 0 < u < t. \\ 0 & \text{otherwise} \end{cases}$$

The joint density of their order statistics $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ is

$$h(u_1, u_2, \ldots, u_n) = n! f(u_1) f(u_2) \ldots f(u_n) = n!(1/t)^n$$

for $0 \le u_1 \le u_2 \le \ldots \le u_n < t$ and 0 elsewhere.

Theorem 5.2

Given $N(t) = n$,

$$(S_1, S_2, \ldots, S_n) \sim (U_{(1)}, U_{(2)}, \ldots, U_{(n)})$$

where $(U_{(1)}, \ldots, U_{(n)})$ are the order statistics of $(U_1, \ldots, U_n) \sim$ i.i.d Uniform $(0, t)$, i.e., the joint conditional density of $S_1, S_2, \ldots$, S_n is

$$f(s_1, s_2, \ldots, s_n | N(t) = n) = n!/t^n, \ 0 < s_1 < s_2 < \ldots < s_n$$

Proof. The event that $S_1 = s_1$, $S_2 = s_2, \ldots$, $S_n = s_n$, $N(t) = n$ is equivalent to the event $T_1 = s_1$, $T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$. Hence, by Proposition 5.1, we have the conditional joint density of $S_1, \ldots, S_n$ given $N(t) = n$ as follows:

$$\begin{aligned} f(s_1, \ldots, s_n | N(t) = n) &= \frac{f(s_1, \ldots, s_n, N(t) = n)}{\mathrm{P}(N(t) = n)} \\ &= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \ldots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{e^{-\lambda t}(\lambda t)^n / n!} \\ &= n! t^{-n}, \quad 0 < s_1 < \ldots < s_n < t \end{aligned}$$

Example 5.21. Insurance claims comes according to a Poisson process $\{N(t)\}$ with rate λ. Let

- S_i = the time of the ith claims
- C_i = amount of the ith claims, i.i.d with mean μ, indep. of $\{N(t)\}$

Then the total discounted cost by time t at discount rate α is given by

$$D(t) = \sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}.$$

Then

$$\begin{aligned}\mathbb{E}[D(t)|N(t)] &= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}\middle| N(t)\right] \overset{(5.2)}{=} \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_{(i)}}\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_i}\right] = \sum_{i=1}^{N(t)} \mathbb{E}[C_i]\mathbb{E}\left[e^{-\alpha U_i}\right] \\ &= N(t)\mu \int_0^t \frac{1}{t} e^{-\alpha x} dx = N(t)\frac{\mu}{\alpha t}(1-e^{-\alpha t})\end{aligned}$$

Thus $\mathbb{E}[D(t)] = \mathbb{E}[N(t)]\frac{\mu}{\alpha t}(1-e^{\alpha t}) = \frac{\lambda\mu}{\alpha}(1-e^{-\alpha t})$

Superposition

The sum of two independent Poisson processes with respective rates λ_1 and λ_2, called the **superposition** of the processes, is again a Poisson process but with rate $\lambda_1 + \lambda_2$.

The proof is straight forward from Definition 5.3 and hence omitted.

Remark: By repeated application of the above arguments we can see that the superposition of k independent Poisson processes with rates $\lambda_1, \cdots, \lambda_k$ is again a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.

Thinning

Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ.
At each arrival of events, it is classified as a

$$\begin{cases} \text{Type 1 event with probability} & p \quad \text{or} \\ \text{Type 2 event with probability} & 1-p, \end{cases}$$

independently of all other events. Let

$$N_i(t) = \# \text{ of type } i \text{ events occurred during } [0,t],\ i = 1,2.$$

Note that $N(t) = N_1(t) + N_2(t)$.

Proposition 5.2

$\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$.

Furthermore, the two processes are independent.

Proof of Proposition 5.2

First observe that given $N(t) = n + m$,

$$N_1(t) \sim Binomial(n+m, p). \qquad \text{(why?)}$$

Thus $\mathrm{P}(N_1(t) = n, N_2(t) = m)$

$$= \mathrm{P}(N_1(t) = n, N_2(t) = m | N(t) = n+m)\mathrm{P}(N(t) = n+m)$$

$$= \binom{n+m}{n} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!}$$

$$= e^{-\lambda t p} \frac{(\lambda p t)^n}{n!} e^{-\lambda t(1-p)} \frac{(\lambda(1-p)t)^m}{m!}$$

$$= \mathrm{P}(N_1(t) = n)\mathrm{P}(N_2(t) = m).$$

This proves the independence of $N_1(t)$ and $N_2(t)$ and that

$$N_1(t) \sim Poisson(\lambda p t), \quad N_2(t) \sim Poisson(\lambda(1-p)t).$$

Both $\{N_1(t)\}$ and $\{N_2(t)\}$ inherit the stationary and independent increment properties from $\{N(t)\}$, and hence are both Poisson processes.

Some "Converse" of Thinning & Superposition

Consider two indep. Poisson processes $\{N_A(t)\}$ and $\{N_B(t)\}$ w/ respective rates λ_A and λ_B. Let

$$S_n^A = \text{arrival time of the } n\text{th } A \text{ event}$$
$$S_m^B = \text{arrival time of the } m\text{th } B \text{ event}$$

Find $\mathrm{P}(S_n^A < S_m^B)$.

Approach 1:
Observer that $S_n^A \sim Gamma(n, \lambda_A)$, $S_m^B \sim Gamma(m, \lambda_B)$ and they are independent. Thus

$$\mathrm{P}(S_n^A < S_m^B) = \int_{x<y} \lambda_A e^{-\lambda_A x} \frac{(\lambda_A x)^{n-1}}{(n-1)!} \lambda_B e^{-\lambda_B y} \frac{(\lambda_B y)^{m-1}}{(m-1)!} dx dy$$

Some "Converse" of Thinning & Superposition (Cont'd)

Let $N(t) = N_A(t) + N_B(t)$ be the superposition of the two processes. Let

$$I_i = \begin{cases} 1 & \text{if the } i\text{th event in the superpositon process is an } A \text{ event} \\ 0 & \text{otherwise} \end{cases}.$$

The I_i, $i = 1, 2, \ldots$ are i.i.d. Bernoulli(p), where $p = \dfrac{\lambda_A}{\lambda_A + \lambda_B}$.

Approach 2:

$$\mathrm{P}(S_n^A < S_1^B) = \mathrm{P}(\text{the first } n \text{ events are all } A \text{ events}) = \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^n$$

$$\begin{aligned} &\mathrm{P}(S_n^A < S_m^B) = \mathrm{P}(\text{at least } n \ A \text{ events occur before } m \ B \text{ events}) \\ &= \mathrm{P}(\text{at least } n \text{ heads before } m \text{ tails}) \\ &= \mathrm{P}(\text{at least } n \text{ heads in the first } n+m-1 \text{ tosses}) \\ &= \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^k \left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right)^{n+m-1-k} \end{aligned}$$

Proposition 5.3 (Generalization of Proposition 5.2)

Consider a Poisson process with rate λ. If an event occurs at time t will be classified as a type i event with probability $p_i(t)$, $i = 1, \ldots, k$, $\sum_i p_i(t) = 1$, for all t, independently of all other events. then

$$N_i(t) = \text{number of type } i \text{ events occurring in } [0, t],\ i = 1, \ldots, k.$$

Note $N(t) = \sum_{i=1}^{k} N_i(t)$. Then $N_i(t)$, $i = 1, \ldots, k$ are independent Poisson random variables with means $\lambda \int_0^t p_i(s) ps$.

Remark: Note $\{N_i(t), t \geq 0\}$ are NOT Poisson processes.

Example

- Policyholders of a certain insurance company have accidents occurring according to a Poisson process with rate λ.
- The amount of time T from when the accident occurs until a claim is made has distribution $G(t) = \mathrm{P}(T \leq t)$.
- Let $N_c(t)$ be the number of claims made by time t.

Find the distribution of $N_c(t)$.

Solution. Suppose an accident occurred at time s. It is claimed by time t if $s + T \leq t$, i.e., with probability

$$p(s) = \mathrm{P}(T \leq t - s) = G(t - s).$$

We call an accident type I if it's completed before t, and type II otherwise. By Proposition 5.3, $N_c(t)$ has a Poisson distribution with mean

$$\lambda \int_0^t p(s) ps = \lambda \int_0^t G(t - s) ds = \lambda \int_0^t G(s) ds$$

5.4.1 Nonhomogeneous Poisson Process

Definition 5.4a. A nonhomogeneous (a.k.a. non-stationary) Poisson process with intensity function $\lambda(t) \geq 0$ is a counting process $\{N(t), t \geq 0\}$ satisfying

(i) $N(0) = 0$.

(ii) having independent increments.

(iii) $\mathrm{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$.

(iv) $\mathrm{P}(N(t+h) - N(t) \geq 2) = o(h)$.

Definition 5.4b. A nonhomogeneous Poisson process with intensity function $\lambda(t) \geq 0$ is a counting process $\{N(t), t \geq 0\}$ satisfying

(i) $N(0) = 0$,

(ii) for $s, t \geq 0$, $N(t+s) - N(s)$ is independent of $N(s)$ (independent increment)

(iii) For $s, t \geq 0$, $N(t+s) - N(s) \sim Poisson(m(t+s) - m(s))$, where $m(t) = \int_0^t \lambda(u)du$

The two definitions are equivalent.

The Interarrival Times of a Nonhomogeneous Poisson Process Are NOT Independent

A nonhomogeneous Poisson process **has independent increment** but its **interarrival times** between events are

- neither independent
- nor identically distributed.

Proof. Homework.

Proposition 5.4

Let $\{N_1(t), t \geq 0\}$, and $\{N_2(t), t \geq 0\}$ be two independent nonhomogeneous Poisson process with respective intensity functions $\lambda_1(t)$ and $\lambda_2(t)$, and let $N(t) = N_1(t) + N_2(t)$. Then

(a) $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda_1(t) + \lambda_2(t)$.

(b) Given that an event of the $\{N(t), t \geq 0\}$ process occurs at time t then, independent of what occurred prior to t , the event at t was from the $\{N_1(t)\}$ process with probability

$$\frac{\lambda_1(t)}{\lambda_1(t) + \lambda_2(t)}.$$

5.4.2 Compound Poisson Processes

Definition. Let $\{N(t)\}$ be a (homogeneous) Poisson process with rate λ and Y_1, $Y_2, \ldots$ are i.i.d random variables independent of $\{N(t)\}$. The process

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

is called a *compound Poisson process*, in which $X(t)$ is defined as 0 if $N(t) = 0$.

A compound Poisson process has

- **independent increment**, since $X(t+s) - X(s) = \sum_{i=1}^{N(t+s)-N(s)} Y_{i+N(s)}$ is independent of $X(s) = \sum_{i=1}^{N(s)} Y_i$, and
- **stationary increment**, since $X(t+s) - X(s) = \sum_{i=1}^{N(t+s)-N(s)} Y_{i+N(s)}$ has the same distribution as $X(t) = \sum_{i=1}^{N(t)} Y_i$

The Mean of a Compound Poisson Process

Suppose $\mathbb{E}[Y_i] = \mu_Y$, $\mathrm{V}ar(Y_i) = \sigma_Y^2$. Note that $\mathbb{E}[N(t)] = \lambda t$.

$$\begin{aligned}
\mathbb{E}[X(t)|N(t)] &= \sum_{i=1}^{N(t)} \mathbb{E}[Y_i|N(t)] \\
&= \sum_{i=1}^{N(t)} \mathbb{E}[Y_i] \quad \text{(since } Y_i\text{'s are indep. of } N(t)\text{)} \\
&= N(t)\mu_Y
\end{aligned}$$

Thus

$$\mathbb{E}[X(t)] = \mathbb{E}[\mathbb{E}[X(t)|N(t)]] = \mathbb{E}[N(t)]\mu_Y = \lambda t \mu_Y$$

Variance of a Compound Poisson Process (Cont'd)

Similarly, using that $\mathbb{E}[N(t)] = \mathrm{V}ar(N(t)) = \lambda t$, we have

$$\begin{aligned}
\mathrm{V}ar[X(t)|N(t)] &= \mathrm{V}ar\left(\sum_{i=1}^{N(t)} Y_i \middle| N(t)\right) \\
&= \sum_{i=1}^{N(t)} \mathrm{V}ar(Y_i|N(t)) \\
&= \sum_{i=1}^{N(t)} \mathrm{V}ar(Y_i) \quad \text{(since } Y_i\text{'s are indep. of } N(t)\text{)} \\
&= N(t)\sigma_Y^2 \\
\mathbb{E}[\mathrm{V}ar(X(t)|N(t))] &= \mathbb{E}[N(t)\sigma_Y^2] = \lambda t \sigma_Y^2 \\
\mathrm{V}ar(\mathbb{E}[X(t)|N(t)]) &= \mathrm{V}ar(N(t)\mu_Y) = \mathrm{V}ar(N(t))\mu_Y^2 = \lambda t \mu_Y^2
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{V}ar(X(t)) &= \mathbb{E}[\mathrm{V}ar[X(t)|N(t)]] + \mathrm{V}ar(\mathbb{E}[X(t)|N(t)]) \\
&= \lambda t(\sigma_Y^2 + \mu_Y^2) = \lambda t \mathbb{E}[Y_i^2]
\end{aligned}$$

CLT of a Compound Poisson Process

As $t \to \infty$, the distribution of

$$\frac{X(t) - \mathbb{E}[X(t)]}{\sqrt{\mathrm{V}ar(X(t))}} = \frac{X(t) - \lambda t \mu_Y}{\sqrt{\lambda t(\sigma_Y^2 + \mu_Y^2)}}$$

converges to a standard normal distribution $N(0, 1)$.

5.4.3 Conditional Poisson Processes

Definition. A *conditional* (or *mixed*) *Poisson process* $\{N(t), t \geq 0\}$ is a counting process satisfying

(i) $N(0) = 0$,

(ii) having stationary increment, and

(iii) there is a random variable $\Lambda > 0$ with probability density $g(\lambda)$, such that given $\Lambda = \lambda$,

$$N(t+s) - N(s) \sim Poisson(\lambda t),$$

i.e.,

$$\mathrm{P}(N(t+s) - N(s) = k) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(\lambda) d\lambda, \; k = 0, 1, \ldots$$

Remark: In general, a conditional Poisson process does NOT have independent increment.

$$
\begin{aligned}
&\mathrm{P}(N(s) = j, N(t+s) - N(s) = k) \\
&= \int_0^\infty e^{-\lambda s}\frac{(\lambda s)^j}{j!}e^{-\lambda t}\frac{(\lambda t)^k}{k!}g(\lambda)d\lambda \\
&\neq \left(\int_0^\infty e^{-\lambda s}\frac{(\lambda s)^j}{j!}g(\lambda)d\lambda\right)\left(\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^k}{k!}g(\lambda)d\lambda\right) \\
&= \mathrm{P}(N(s) = j)\mathrm{P}(N(t+s) - N(s) = k)
\end{aligned}
$$