Markov chains (MCs)

Discrete-time time-homogeneous MC

Transition probabilities
- Chapman Kolmogorov equation
- Goal: calculate n step transition probabilities

Accessible
- Communicate
- Communication class
- Irreducible

Period
- GCD of $\{n \geq 1 \mid P^n_{ii} > 0\}$
- Periodic/Aperiodic
- Periodicity is a class property

Recurrence/Transience
- Characterization using n-step transition probabilities
  - Recurrence is a class property
  - All states of a finite irreducible MC are recurrent
- Positive recurrence/Null recurrence (def: expected first return time)
  - Both are class properties
  - All states of a finite irreducible MC are positive recurrent
- Expected first return time

Stationary distribution (solutions to balanced equations)
- 1-d symmetric random walk has no stationary distribution
  - Irreducible MC has (a unique) stationary distribution iff it is positive recurrent.
- May have multiple stationary distributions

Limiting distribution (every row of $\lim_{n} P^{n}$ is the same, and is a prob vector)
- Limiting distribution must be stationary
- May not exist (for periodic Markov chains)
  - When does a MC have limiting distribution?
    - At least we need irreducible, aperiodic

Fundamental limit theorem
- Long run proportion of visit = $1/E[T_j]$
- For irreducible MC, it is positive recurrent iff there exists solutions to the stationary eq.
  - For irreducible and positive recurrent MC, we have $\pi_i = 1 / E[T_i] = $
    - Periodic: long run proportion of time of a state
    - Aperiodic: limit of transition probability
- If it is further aperiodic, then limiting distribution exists and is equal to the stationary distribution

Time reversible MC
- Iff the stationary distribution obeys detailed balanced eq
  - Solutions to detailed balanced eq must be stationary distribution
- Random walk on weighted graph

First step analysis
- Law of total expectation
- Law of total variance
- Examples!

Example: 1-D symmetric random walk
- Countable state space
- Irreducible
- Recurrent
- No stationary distribution — null recurrent

Branching process
- How does mean evolve?
- How does variance evolve?
- What's the probability of extinction?