Branching Processes and Generating Functions

Cong Ma

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Branching Processes

Consider a population of individuals.

- All individuals have the same lifetime
- Each individual will produce a random number of offsprings at the end of its life

Let X_n = size of the n-th generation, $n = 0, 1, 2, \ldots$.
If $X_{n-1} = k$, the k individuals in the $(n-1)$-th generation will independently produce $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,k}$ new offsprings, and $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,X_{n-1}}$ are i.i.d such that

$$P(Z_{n,i} = j) = P_j,\ j \geq 0.$$

We suppose that $P_j < 1$ for all $j \geq 0$.

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i} \tag{1.1}$$

$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \ldots\}$.

Mean of a Branching Process

Let $\mu = \mathsf{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. Since $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathsf{E}[X_n|X_{n-1}] = \mathsf{E}\left[\sum_{i=1}^{X_{n-1}} Z_{n,i}\middle|X_{n-1}\right] = X_{n-1}\mathsf{E}[Z_{n,i}] = X_{n-1}\mu$$

So

$$\mathsf{E}[X_n] = \mathsf{E}[\mathsf{E}[X_n|X_{n-1}]] = \mathsf{E}[X_{n-1}\mu] = \mu\mathsf{E}[X_{n-1}]$$

If $X_0 = 1$, then

$$\mathsf{E}[X_n] = \mu\mathsf{E}[X_{n-1}] = \mu^2\mathsf{E}[X_{n-2}] = \ldots = \mu^n$$

- If $\mu < 1 \Rightarrow \mathsf{E}[X_n] \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} \mathrm{P}(X_n \geq 1) = 0$ the branching processes will eventually die out.
- What if $\mu = 1$ or $\mu > 1$?

Variance of a Branching Process

Let $\sigma^2 = \mathsf{Var}[Z_{n,i}] = \sum_{j=0}^{\infty}(j-\mu)^2 P_j$. $\mathsf{Var}(X_n)$ may be obtained using the conditional variance formula

$$\mathsf{Var}(X_n) = \mathsf{E}[\mathsf{Var}(X_n|X_{n-1})] + \mathsf{Var}(\mathsf{E}[X_n|X_{n-1}]).$$

Again from that $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathsf{E}[X_n|X_{n-1}] = X_{n-1}\mu, \quad \mathsf{Var}(X_n|X_{n-1}) = X_{n-1}\sigma^2$$

and hence

$$\mathsf{Var}(\mathsf{E}[X_n|X_{n-1}]) = \mathsf{Var}(X_{n-1}\mu) = \mu^2 \mathsf{Var}(X_{n-1})$$
$$\mathsf{E}[\mathsf{Var}(X_n|X_{n-1})] = \sigma^2 \mathsf{E}[X_{n-1}] = \sigma^2 \mu^{n-1} \mathsf{E}[X_0].$$

So

$$\begin{aligned}\mathsf{Var}(X_n) &= \sigma^2\mu^{n-1}\mathsf{E}[X_0] + \mu^2\mathsf{Var}(X_{n-1}) \\ &= \sigma^2\mathsf{E}[X_0](\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}) + \mu^{2n}\mathsf{Var}(X_0) \\ &= \begin{cases} \sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right)\mathsf{E}[X_0] + \mu^{2n}\mathsf{Var}(X_0) & \text{if } \mu \neq 1 \\ n\sigma^2\mathsf{E}[X_0] + \mu^{2n}\mathsf{Var}(X_0) & \text{if } \mu = 1 \end{cases}\end{aligned}$$

Extinction Probability of a Branching Process

Define

$$E_n = \{X_n = 0\}, \quad n \geq 1,$$

the event that the population is extinct by generation n, and let

$$E = \{\text{population is ultimately extinct}\}.$$

Then

$$E = \{X_n = 0 \text{ for some } n \geq 1\} = \bigcup_{n=1}^{\infty} E_n.$$

Since

$$E_1 \subseteq E_2 \subseteq \cdots ,$$

it follows that

$$P(E) = P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n) = \lim_{n\to\infty} P(X_n = 0).$$

Extinction Probability of a Branching Process

$$\text{Let } \pi_0 = \lim_{n\to\infty} \mathrm{P}(X_n = 0 | X_0 = 1)$$
$$= \mathrm{P}(\text{the population will eventually die out} | X_0 = 1)$$

Extinction Probability in the Subcritical Case

Extinction by generation n

$$P(X_n = 0) = 1 - P(X_n \geq 1) = 1 - \sum_{k=1}^{\infty} P(X_n = k)$$

$$\geq 1 - \sum_{k=1}^{\infty} k\, P(X_n = k) = 1 - \mathsf{E}(X_n) = 1 - \mu^n.$$

Taking limits

$$P(E) = \lim_{n\to\infty} P(X_n = 0) \geq \lim_{n\to\infty} (1 - \mu^n) = 1, \qquad (\mu < 1).$$

Thus, $P(E) = 1$: a subcritical branching process goes extinct a.s.

Generating Functions

For a non-negative integer-valued random variable T, the generating function of T is the expected value of s^T as a function of s

$$G(s) = \mathsf{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k),$$

in which s^T is defined as 0 if $T = \infty$.
Since $0 \leq \mathrm{P}(T = k) \leq 1$, the generating function is always well-defined for $-1 \leq s \leq 1$

Examples of Generating Functions

- If T has a geometric distribution: $P(T=k)=p(1-p)^k$, $k=0,1,2,\ldots$, the generating function of T is

$$G(s)=\sum_{k=0}^{\infty} s^k \mathrm{P}(T=k)=\sum_{k=0}^{\infty} s^k p(1-p)^k=\frac{p}{1-(1-p)s}$$

- If T has a Binomial distribution $P(T=k)=\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,1,2,\ldots,n$, the generating function of T is

$$\begin{aligned} G(s)=\sum_{k=0}^{\infty} s^k \mathrm{P}(T=k) &= \sum_{k=0}^{\infty} s^k \binom{n}{k} p^k(1-p)^{n-k} \\ &= (ps+(1-p))^n \end{aligned}$$

Properties of Generating Function

$$G(s) = \mathsf{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k)$$

- $G(s)$ is a power series converging absolutely for all $-1 \le s \le 1$. since $0 \le \mathrm{P}(T = k) \le 1$ and $\sum_k \mathrm{P}(T = k) \le 1$.

- $G(1) = \mathrm{P}(T < \infty) \begin{cases} = 1 & \text{if } T \text{ is finite w/ prob. 1} \\ < 1 & \text{otherwise} \end{cases}$

- $P(T = k) = \dfrac{G^{(k)}(0)}{k!}$

  Knowing $G(s) \Leftrightarrow$ Knowing $\mathrm{P}(T = k)$ for all $k = 0, 1, 2, \ldots$

More Properties of Generating Functions

$$G(s) = \mathsf{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k)$$

- $\mathsf{E}[T] = \lim_{s \to 1^-} G'(s)$ if it exists because

$$G'(s) = \frac{d}{ds}\mathsf{E}[s^T] = \mathsf{E}[Ts^{T-1}] = \sum_{k=1}^{\infty} s^{k-1} k \mathrm{P}(T = k).$$

- $\mathsf{E}[T(T-1)] = \lim_{s \to 1^-} G''(s)$ if it exists because

$$G''(s) = \mathsf{E}[T(T-1)s^{T-2}] = \sum_{k=2}^{\infty} s^{k-2} k(k-1) \mathrm{P}(T = k)$$

- If T and U are **independent** non-negative-integer-valued random variables, with generating function $G_T(s)$ and $G_U(s)$ respectively, then the generating function of $T + U$ is

$$G_{T+U}(s) = \mathsf{E}[s^{T+U}] = \mathsf{E}[s^T]\mathsf{E}[s^U] = G_T(s)G_U(s)$$

Properties of Probability Generating Function

Definition

Let

$$G(s) = \mathsf{E}(s^X)$$

be the probability generating function (PGF) of a discrete random variable X.

1. Then:
   a. $G(1) = 1$,
   b. $P(X = k) = \dfrac{G^{(k)}(0)}{k!}, \quad k \geq 0,$
   c. $\mathsf{E}(X) = G'(1)$,
   d. $\mathsf{Var}(X) = G''(1) + G'(1) - (G'(1))^2$.
2. If $G_X(s) = G_Y(s)$ for all s, then X and Y have the same distribution.
3. If X and Y are independent, then

$$G_{X+Y}(s) = G_X(s)G_Y(s).$$

Generating Functions of the Branching Processes

Let $g(s) = \mathsf{E}[s^{Z_{n,i}}] = \sum_{k=0}^{\infty} P_k s^k$ be the generating function of $Z_{n,i}$, and $G_n(s)$ be the generating function of X_n, $n = 0, 1, 2, \ldots$. Then $\{G_n(s)\}$ satisfies the following two iterative equations.

$$\text{(i) } G_{n+1}(s) = G_n(g(s)) \quad \text{for } n = 0, 1, 2, \ldots$$

$$\text{(ii) } G_{n+1}(s) = g(G_n(s)) \quad \text{if } X_0 = 1, \text{ for } n = 0, 1, 2, \ldots$$

Proof of (i).

$$\begin{aligned}
\mathsf{E}[s^{X_{n+1}}|X_n] = \mathsf{E}\left[s^{\sum_{i=1}^{X_n} Z_{n,i}}\right] &= \mathsf{E}\Big[\prod_{i=1}^{X_n} s^{Z_{n,i}}\Big] \\
&= \prod_{i=1}^{X_n} \mathsf{E}[s^{Z_{n,i}}] \quad \text{by indep. of } Z_{n,i}\text{'s} \\
&= \prod_{i=1}^{X_n} g(s) \quad \text{as } g(s) = \mathsf{E}[s^{Z_{n,i}}] \\
&= g(s)^{X_n}
\end{aligned}$$

From which, we have

$$G_{n+1}(s) = \mathsf{E}[s^{X_{n+1}}] = \mathsf{E}[\mathsf{E}[s^{X_{n+1}}|X_n]] = \mathsf{E}[g(s)^{X_n}] = G_n(g(s))$$

since $G_n(s) = \mathsf{E}[s^{X_n}]$.

Proof of (ii) $G_{n+1}(s) = g(G_n(s))$ if $X_0 = 1$

Suppose there are k individuals in the first generation ($X_1 = k$). Let Y_i be the number offspring of the ith individual in the first generation in the $(n+1)$st generation. Obviously,

$$X_{n+1} = Y_1 + \ldots + Y_k.$$

Observe $Y_1, \ldots, Y_k$'s are indep and each has the same distn. as X_n since they are all the size of the nth generation of a single ancestor. Thus, by indep. of Y_i's

$$\mathsf{E}[s^{X_{n+1}}|X_1 = k] = \mathsf{E}\big[s^{Y_1+\ldots+Y_k}\big] = \mathsf{E}\left[\prod_{i=1}^{k} s^{Y_i}\right] = \prod_{i=1}^{k} \mathsf{E}[s^{Y_i}]$$

Since Y_i's have the same dist'n as X_n and $G_n(s) = \mathsf{E}[s^{X_n}]$, we have

$$\mathsf{E}[s^{X_{n+1}}|X_1 = k] = \prod_{i=1}^{k} G_n(s) = (G_n(s))^k$$

Since $X_0 = 1$, $X_1 = Z_{1,1}$, and hence $\mathrm{P}(X_1 = k) = P_k$.

$$G_{n+1}(s) = \mathsf{E}[s^{X_{n+1}}] = \sum_{k=0}^{\infty} \mathsf{E}[s^{X_{n+1}}|X_1 = k]P_k = \sum_{k=0}^{\infty} (G_n(s))^k P_k = g(G_n(s)),$$

where the last equality comes from that $g(s) = \sum_{k=0}^{\infty} P_k s^k$.

Example: calculating distributions of X_n

Suppose $X_0 = 1$, and $(P_0, P_1, P_2) = (1/4, 1/2, 1/4)$. Find the distribution of X_2.

Sol.

$$g(s) = \frac{1}{4}s^0 + \frac{1}{2}s^1 + \frac{1}{4}s^2 = (1+s)^2/4.$$

Since $X_0 = 1$, $G_0(s) = \mathsf{E}[s^{X_0}] = \mathsf{E}[s^1] = s$. From (i) we have

$$\begin{aligned} G_1(s) &= G_0(g(s)) = g(s) = (1+s)^2/4 \\ G_2(s) &= G_1(g(s)) = \frac{1}{4}(1 + \frac{1}{4}(1+s)^2)^2 = \frac{1}{64}(5 + 2s + s^2)^2 \\ &= \frac{1}{64}(25 + 20s + 14s^2 + 4s^3 + s^4) = \sum_{k=0}^{\infty} \mathrm{P}(X_2 = k)s^k \end{aligned}$$

The coefficient of s^k in the polynomial of $G_2(s)$ is the chance that $X_2 = k$.

k	0	1	2	3	4
$\mathrm{P}(X_2 = k)$	$\frac{25}{64}$	$\frac{20}{64}$	$\frac{14}{64}$	$\frac{4}{64}$	$\frac{1}{64}$

and $\mathrm{P}(X_2 = k) = 0$ for $k \geq 5$.

Extinction Probability of a Branching Process

$$\text{Let } \pi_0 = \lim_{n\to\infty} \mathrm{P}(X_n = 0 | X_0 = 1)$$
$$= \mathrm{P}(\text{the population will eventually die out} | X_0 = 1)$$

As $G_n(s) = \mathsf{E}[s^{X_n}] = \sum_{k=0}^{\infty} P(X_n = k)s^k$, plugging in $s = 0$, we get

$$G_n(0) = P(X_n = 0) = P(\text{extinct by the } n\text{th generation}).$$

Recall that if $X_0 = 1$, $G_1(s) = g(s)$, and $G_{n+1}(s) = g(G_n(s))$. We can compute $G_n(0)$ iteratively as follows

$$G_1(0) = g(0)$$
$$G_{n+1}(0) = g(G_n(0)), \quad n = 1, 2, 3, \ldots$$

Finally, we can get the extinction probability by taking the limit

$$\pi_0 = \lim_{n\to\infty} G_n(0).$$

Extinction Probability

Theorem 1.1 (Extinction Probability)

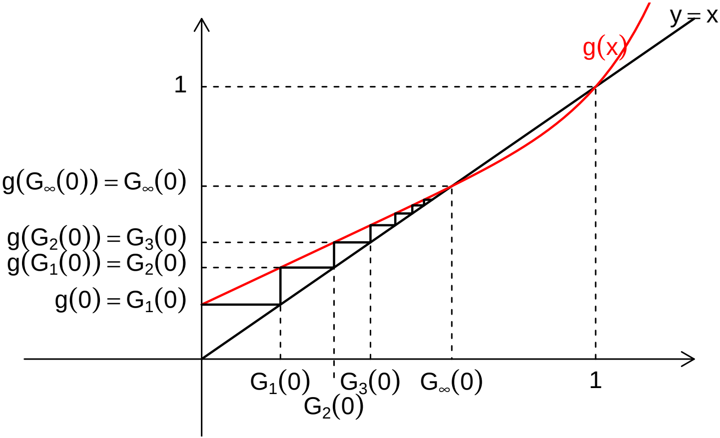
Given a branching process, let G be the probability generating function of the offspring distribution.

Then the probability of eventual extinction is the smallest positive root of the equation

$$s = G(s).$$

If $\mu \leq 1$ (the subcritical and critical cases), then the extinction probability is equal to 1.

Proof of Part 1



Proof of Extinction Probability (Step 1)

Let

$$e_n = P(Z_n = 0)$$

be the probability that the population is extinct by generation n.

Using the branching property and PGFs,

$$e_n = P(Z_n = 0) = G_n(0) = G(G_{n-1}(0)) = G(e_{n-1}), \qquad n \geq 1.$$

Recursive relation

$$\boxed{e_n = G(e_{n-1})}$$

Proof Sketch: Fixed Point Iteration

Let $e_n := \mathrm{P}(X_n = 0 \mid X_0 = 1) = G_n(0)$.
If $X_0 = 1$, we have the PGF recursion

$$G_{n+1}(s) = g(G_n(s)).$$

Plugging in $s = 0$ gives

$$e_{n+1} = G_{n+1}(0) = g(G_n(0)) = g(e_n).$$

Key recursion

$$\boxed{e_{n+1} = g(e_n)}$$

Proof of Extinction Probability (Step 2)

From earlier results,

$$e_n \to e \quad \text{as } n \to \infty,$$

where e is the eventual extinction probability.

Taking limits in

$$e_n = G(e_{n-1})$$

and using continuity of G, we obtain

$$e = G(e).$$

Conclusion

The extinction probability e is a root of

$$s = G(s).$$

Proof of Extinction Probability (Step 3)

Let $x > 0$ be any solution of

$$x = G(x).$$

We will show that

$$e \leq x.$$

Recall that

$$G(s) = \sum_{k=0}^{\infty} s^k P(X = k)$$

is an increasing function on $(0, 1]$.

Proof of Extinction Probability (Step 4)

Base case:

$$e_1 = P(Z_1 = 0) = G(0) \leq G(x) = x.$$

Inductive step:
Assume

$$e_k \leq x \quad \text{for all } k < n.$$

Then

$$e_n = G(e_{n-1}) \leq G(x) = x.$$

Proof of Extinction Probability (Conclusion)

Since

$$e_n \leq x \quad \text{for all } n,$$

taking limits gives

$$e \leq x.$$

Result

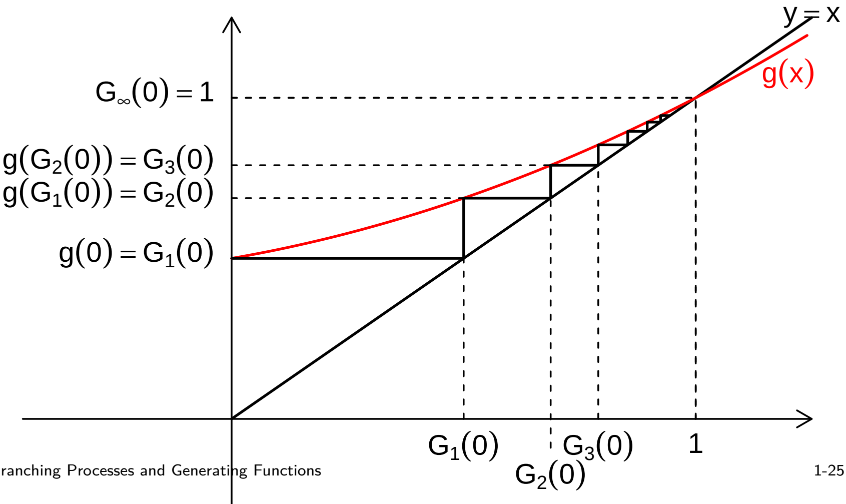
The extinction probability e is the smallest positive solution of

$$s = G(s).$$

Proof of Part 2

Let $\mu = \mathsf{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. If $\mu \le 1$, the extinction probability π_0 is 1.

Proof.



Formal Proof

Let $h(s) = g(s) - s$. Since $g(1) = 1$, $g'(1) = \mu$,

$$h(1) = g(1) - 1 = 0,$$

$$h'(s) = \Big(\sum_{j=1}^{\infty} jP_j s^{j-1}\Big) - 1 \le \Big(\sum_{j=1}^{\infty} jP_j\Big) - 1 = \mu - 1 \quad \text{for } 0 \le s < 1$$

Thus $\mu \le 1 \Rightarrow h'(s) \le 0$ for $0 \le s < 1$

$\Rightarrow h(s)$ is non-increasing in $[0, 1)$

$\Rightarrow h(s) > h(1) = 0$ for $0 \le s < 1$

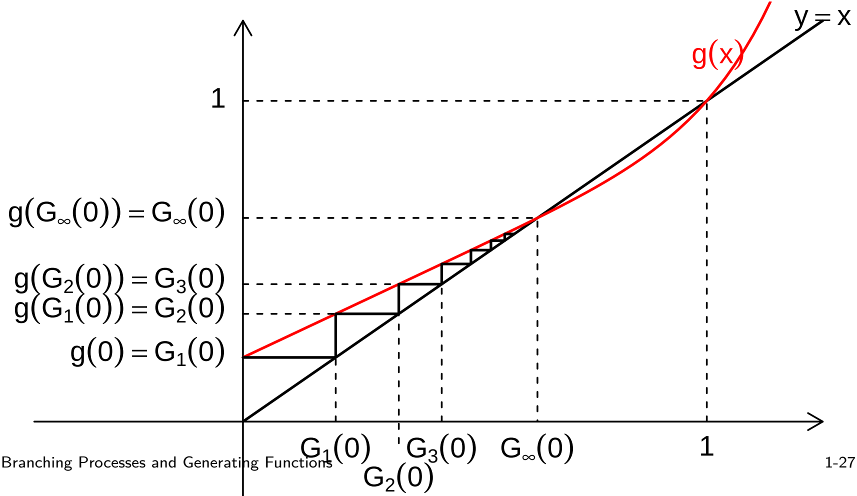
$\Rightarrow g(s) > s$ for $0 \le s < 1$

$\Rightarrow$ There is no root in [0,1).

Extinction Probability When $\mu > 1$

If $\mu > 1$, there is a unique root of the equation $g(s) = s$ in the domain $[0, 1)$, and that is the extinction probability.

Proof.



Formal Proof

Let $h(s) = g(s) - s$. Observe that

$$\begin{aligned} h(0) &= g(0) = P_0 > 0 \\ h'(0) &= g'(0) - 1 = P_1 - 1 < 0 \\ \text{Then } \mu > 1 &\Rightarrow h'(1) = \mu - 1 > 0 \\ &\Rightarrow h(s) \text{ is increasing near 1} \\ &\Rightarrow h(1-\delta) < h(1) = 0 \text{ for } \delta > 0 \text{ small enough} \end{aligned}$$

Since $h(s)$ is continuous in $[0, 1)$, there must be a root to $h(s) = s$. The root is unique since

$$h''(s) = g''(s) = \sum_{j=2}^{\infty} j(j-1)P_j s^{j-2} \geq 0 \quad \text{for } 0 \leq s < 1$$

$h(s)$ is convex in [0,1).