Discrete-Time Markov Chains

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Outline

- **What is a Markov chain?**
  - State space, Markov property
  - Stationary chains and transition matrix P
- **Examples**
  - Random walk, Gambler's ruin, Ehrenfest model (and i.i.d. as a degenerate case)
- **Basic computations**
  - n-step transition probabilities $P^{(n)}$
  - Chapman–Kolmogorov equations and matrix powers $P^{(n)} = P^n$
  - $\pi_n(j) = \mathbb{P}(X_n = j)$ and $\pi_n = \pi_0 P^n$

Definitions of DTMC

Consider a stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$ taking values in a finite or countable set $\mathcal{X}$.

- $\mathcal{X}$ is called the **state space**
- If $X_n = i$, $i \in \mathcal{X}$, we say the process is in state i at time n
- Since $\mathcal{X}$ is countable, we can label states by integers (e.g. $\{0, 1, 2, \ldots\}$, $\mathbb{Z}$, or $\{0, \ldots, n\}$ depending on the model).

Definition

A stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$ is called a **Markov chain** if it has the following property:

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_2 = i_2, X_1 = i_1, X_0 = i_0)$$
$$= P(X_{n+1} = j | X_n = i)$$

for all states i_0, i_1, $i_2, \ldots, i_{n-1}$, i, $j \in \mathcal{X}$ and $n \geq 0$.

Transition Probability Matrix

If $P(X_{n+1} = j | X_n = i) = P_{ij}$ does not depend on n, then the process $\{X_n\}$ is called a **stationary Markov chain**. From now on, we consider stationary Markov chains only.

$\{P_{ij}\}$ is called the **transition probabilities**.

$$\mathbb{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i0} & P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

is called the **transition probability matrix**.

Naturally, the transition probabilities $\{P_{ij}\}$ satisfy:

- $P_{ij} \geq 0$ for all i, j
- **Row sums are 1:** $\sum_j P_{ij} = 1$ for all i.

In other words, $\mathbb{P}\,\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, 1, \cdots, 1, \cdots)^T$.

Random Walk

Consider the following random walk on integers

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } 1 - p \end{cases}$$

This is a Markov chain because given $X_n, X_{n-1}, X_{n-2}, \ldots$, the distribution of X_{n+1} depends only on X_n but not $X_{n-1}, X_{n-2}, \ldots$. The state space is

$$\mathcal{X} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} = \mathbb{Z} = \text{all integers}$$

The transition probability is

$$P_{ij} = \begin{cases} p & \text{if } j = i + 1 \\ 1 - p & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

Gambler's Ruin

In each round of a gambling game a player either wins \$1, with probability p, or loses \$1, with probability $1 - p$. The gambler starts with $\$k$. The game stops when the player either loses all their money, or gains a total of $\$n$ $(n > k)$.

The gambler's successive fortunes form a Markov chain on $\{0, 1, \ldots, n\}$ with $X_0 = k$ and transition matrix given by

$$P_{ij} = \begin{cases} p, & \text{if } j = i + 1, \quad 0 < i < n, \\ 1 - p, & \text{if } j = i - 1, \quad 0 < i < n, \\ 1, & \text{if } i = j = 0, \text{ or } i = j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Transition Matrix

Here is the transition matrix with $n = 6$ and $p = \frac{1}{3}$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Gambler's ruin is an example of *simple random walk with absorbing boundaries*. Since $P_{00} = P_{nn} = 1$, when the chain reaches 0 or n, it stays there forever.

Ehrenfest Diffusion Model

Two containers A and B, containing a sum of K balls. At each stage, a ball is selected at random from the totality of K balls, and move to the other container. Let

$X_0 = \#$ of balls in container A in the beginning

$X_n = \#$ of balls in container A after n movements, $n = 1, 2, 3, \ldots$

$$\mathcal{X} = \{0, 1, 2, \ldots, K\}$$

$$P_{ij} = \begin{cases} \dfrac{i}{K} & \text{if } j = i - 1 \\ \dfrac{K - i}{K} & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

IID sequence

An independent and identically distributed sequence of random variables is trivially a Markov chain. Assume that $X_0, X_1, \ldots$ is an i.i.d. sequence that takes values in $\{1, \ldots, k\}$ with

$$\mathbb{P}(X_n = j) = p_j, \quad \text{for } j = 1, \ldots, k \text{ and } n \geq 0,$$

where

$$p_1 + \cdots + p_k = 1.$$

By independence,

$$\mathbb{P}(X_1 = j \mid X_0 = i) = \mathbb{P}(X_1 = j) = p_j.$$

The transition matrix is

$$P = \begin{pmatrix} p_1 & p_2 & \cdots & p_k \\ p_1 & p_2 & \cdots & p_k \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_2 & \cdots & p_k \end{pmatrix}.$$

Joint Distribution of Random Variables in a Markov Chain

Suppose $\{X_n : \ n = 0, 1, 2, \ldots\}$ is a stationary Markov chain with

- state space $\mathcal{X}$ and
- transition probabilities $\{P_{ij} : i, j \in \mathcal{X}\}$.

Define $\pi_0(i) = \mathbb{P}(X_0 = i)$, $i \in \mathcal{X}$ to be the distribution of X_0.

What is the joint distribution of X_0, X_1, X_2?

Joint Distribution

$$\begin{aligned}
&\mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2) \\
&= \mathbb{P}(X_0 = i_0)\mathbb{P}(X_1 = i_1 | X_0 = i_0)\mathbb{P}(X_2 = i_2 | X_1 = i_1, X_0 = i_0) \\
&= \mathbb{P}(X_0 = i_0)\mathbb{P}(X_1 = i_1 | X_0 = i_0)\mathbb{P}(X_2 = i_2 | X_1 = i_1) \quad \text{(Markov)} \\
&= \pi_0(i_0) P_{i_0 i_1} P_{i_1 i_2}
\end{aligned}$$

In general,

$$\begin{aligned}
&\mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}, X_n = i_n) \\
&= \pi_0(i_0) P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}
\end{aligned}$$

n-Step Transition Probabilities

Suppose $\{X_n\}$ is a stationary Markov chain with state space $\mathcal{X}$. Define the n-step transition probabilities

$$P_{ij}^{(n)} = \mathbb{P}(X_{n+k} = j \mid X_k = i) \quad \text{for } i, j \in \mathcal{X} \text{ and } n, k = 0, 1, 2, \ldots$$

How to calculate $P_{ij}^{(n)}$?

Chapman-Kolmogorov Equations

Suppose $\{X_n\}$ is a stationary Markov chain with state space $\mathcal{X}$. Define the n-step transition probabilities

$$P_{ij}^{(n)} = \mathbb{P}(X_{n+k} = j | X_k = i) \quad \text{for } i, j \in \mathcal{X} \text{ and } n, k = 0, 1, 2, \ldots$$

Then for all m, $n \geq 1$,

$$P_{ij}^{(m+n)} = \sum_{k \in \mathcal{X}} P_{ik}^{(m)} P_{kj}^{(n)}$$

Proof

$$
\begin{aligned}
P_{ij}^{(m+n)} &= \mathbb{P}(X_{m+n} = j | X_0 = i) \\
&= \sum_{k \in \mathcal{X}} \mathbb{P}(X_{m+n} = j, X_m = k | X_0 = i) \\
&= \sum_{k \in \mathcal{X}} \mathbb{P}(X_m = k | X_0 = i) \mathbb{P}(X_{m+n} = j | X_m = k, X_0 = i) \\
&= \sum_{k \in \mathcal{X}} \mathbb{P}(X_m = k | X_0 = i) \mathbb{P}(X_{m+n} = j | X_m = k) \quad \text{(Markov)} \\
&= \sum_{k \in \mathcal{X}} P_{ik}^{(m)} P_{kj}^{(n)}
\end{aligned}
$$

Chapman-Kolmogorov Equation in Matrix Notation

For $n = 1, 2, 3, \ldots$, let

$$P^{(n)} = \begin{pmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots & P_{0j}^{(n)} & \cdots \\ P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \cdots & P_{1j}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i0}^{(n)} & P_{i1}^{(n)} & P_{i2}^{(n)} & \cdots & P_{ij}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

be the n-step transition probability matrix.
The Chapman-Kolmogorov equation just asserts that

$$P^{(m+n)} = P^{(m)} \times P^{(n)}$$

Note $P^{(1)} = P$, $\Rightarrow P^{(2)} = P^{(1)} \times P^{(1)} = P \times P = P^2$.
By induction,

$$P^{(n)} = P^{(n-1)} \times P^{(1)} = P^{n-1} \times P = P^n$$

Distribution of X_n

Define $\pi_n(i) = \mathbb{P}(X_n = i)$, $i \in \mathcal{X}$ to be the marginal distribution of X_n, $n = 1, 2, \ldots$. Then again by the law of total probabilities,

$$\begin{aligned} \pi_n(j) &= \mathbb{P}(X_n = j) \\ &= \sum_{k \in \mathcal{X}} \mathbb{P}(X_0 = k)\mathbb{P}(X_n = j | X_0 = k) \qquad (1.1) \\ &= \sum_{k \in \mathcal{X}} \pi_0(k) P_{kj}^{(n)} \end{aligned}$$

Suppose the state space $\mathcal{X}$ is $\{0, 1, 2, \ldots\}$.
If we write the marginal distribution of X_n as a row vector

$$\pi_n = (\pi_n(0), \pi_n(1), \pi_n(2), \ldots),$$

then equation (1.1) is equivalent to

$$\pi_n = \pi_0 P^{(n)} = \pi_0 P^n$$

Ehrenfest Model with 4 Balls

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 4/4 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 4/4 & 0 \end{pmatrix} \end{array}$$

Q1 Find $P_{4,2}^{(4)} = \mathbb{P}(X_4 = 2 | X_0 = 4)$.

Q2 Find $P_{4,2}^{(10)} = \mathbb{P}(X_{10} = 2 | X_0 = 4)$.

Q3 Given $\mathbb{P}(X_0 = i) = 1/5$ for $i = 0, 1, 2, 3, 4$, find $\mathbb{P}(X_4 = 2)$

Q4 Find $\mathbb{P}(X_{10} = 2, X_k \geq 2,$ for $1 \leq k \leq 9 | X_0 = 4)$

Example: Ehrenfest Model, 4 Balls (Cont'd)

To find $P_{4,2}^{(10)}$ for Q4, it's awful lots of work to compute P^{10}...

There are ways to save some work. By the C-K equation,

$$P_{4,2}^{(10)} = \underbrace{P_{4,0}^{(5)} P_{0,2}^{(5)}}_{=0} + P_{4,1}^{(5)} P_{1,2}^{(5)} + \underbrace{P_{4,2}^{(5)} P_{2,2}^{(5)}}_{=0} + P_{4,3}^{(5)} P_{3,2}^{(5)} + \underbrace{P_{4,4}^{(5)} P_{4,2}^{(5)}}_{=0}$$

because it's impossible to move between even states in odd number of moves.

We just need to find $P_{4,1}^{(5)}$, $P_{4,3}^{(5)}$, $P_{1,2}^{(5)}$, and $P_{3,2}^{(5)}$

Example: Ehrenfest Model, 4 Balls (Cont'd)

$$P^5 = P^2 \times P^3$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 5/8 & 0 & 3/8 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 3/8 & 0 & 5/8 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \end{pmatrix} \end{array} \times \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 5/8 & 0 & 3/8 & 0 \\ 5/32 & 0 & 3/4 & 0 & 3/32 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 3/32 & 0 & 3/4 & 0 & 5/32 \\ 0 & 3/8 & 0 & 5/8 & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & & 0 & & \\ & & 3/4 & & \\ & & 0 & & \\ & & 3/4 & & \\ 0 & 15/32 & 0 & 17/32 & 0 \end{pmatrix} \end{array}$$

$$P_{4,2}^{(10)} = P_{4,1}^{(5)} P_{1,2}^{(5)} + P_{4,3}^{(5)} P_{3,2}^{(5)} = \frac{15}{32} \times \frac{3}{4} + \frac{17}{32} \times \frac{3}{4} = \frac{3}{4}.$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

$$\pi_0 = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}).$$

$$\pi_4 = \pi_0 P^4 = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \begin{pmatrix} 5/32 & 0 & 3/4 & 0 & 3/32 \\ 0 & 17/32 & 0 & 15/32 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 15/32 & 0 & 17/32 & 0 \\ 3/32 & 0 & 3/4 & 0 & 5/32 \end{pmatrix}$$

$$\pi_4(2) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \begin{pmatrix} 3/4 \\ 0 \\ 3/4 \\ 0 \\ 3/4 \end{pmatrix}$$

$$= \frac{1}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot \frac{3}{4} = \frac{9}{20}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

Q6: Find $\mathbb{P}(X_{10} = 2, X_k \geq 2, \text{ for } 1 \leq k \leq 9 | X_0 = 4)$.

Tip: Create another process $\{W_n, n = 0, 1, 2, \ldots\}$ with an absorbing state A

$$W_n = \begin{cases} X_n & \text{if } X_k \geq 2 \text{ for all } k = 0, 1, 2, \ldots, n \\ A & \text{if } X_k < 2 \text{ for some } k \leq n \end{cases}$$

What is the state space of $\{W_n\}$?
Is $\{W_n\}$ a Markov chain?

$$W_{n+1} = \begin{cases} A & \text{if } W_n = A \\ W_n + 1 & \text{with prob. } \frac{4 - W_n}{4} \text{ if } W_n \neq A \\ W_n - 1 & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 3 \text{ or } 4 \\ A & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 2 \end{cases}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

Q6: Find $\mathbb{P}(X_{10} = 2, X_k \geq 2, \text{ for } 1 \leq k \leq 9 | X_0 = 4)$.

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Example: Ehrenfest Model, 4 Balls (Cont'd)

Q6: Find $\mathbb{P}(X_{10} = 2, X_k \geq 2, \text{ for } 1 \leq k \leq 9 | X_0 = 4)$.

Tip: Create another process $\{W_n, n = 0, 1, 2, \ldots\}$ with an absorbing state A

$$W_n = \begin{cases} X_n & \text{if } X_k \geq 2 \text{ for all } k = 0, 1, 2, \ldots, n \\ A & \text{if } X_k < 2 \text{ for some } k \leq n \end{cases}$$

What is the state space of $\{W_n\}$? $\{A, 2, 3, 4\}$
Is $\{W_n\}$ a Markov chain?

$$W_{n+1} = \begin{cases} A & \text{if } W_n = A \\ W_n + 1 & \text{with prob. } \frac{4-W_n}{4} \text{ if } W_n \neq A \\ W_n - 1 & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 3 \text{ or } 4 \\ A & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 2 \end{cases}$$

Yes, $\{W_n\}$ is a Markov chain.

Example: Ehrenfest Model, 4 Balls (Cont'd)

What is the transition probability of $\{W_n\}$?

$$P_W = \begin{array}{c} \\ A \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} A & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2/4 & 0 & 2/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Observe that $\mathbb{P}_{W,i,j}$ equals the transition prob. of the original process $\mathbb{P}_{i,j}$ for $i, j \neq A$.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 4/4 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

How does $\{W_n\}$ helps us to solve Q6?

$$\text{Observe that } \mathbb{P}(X_{10} = 2, X_k \geq 2, \text{ for } 1 \leq k \leq 9 | X_0 = 4)$$
$$= \mathbb{P}(W_{10} = 2 | W_0 = 4) = P_{W,4,2}^{(10)}$$

It's still an awful lot of work to compute $P_{W,4,2}^{(10)}$.

By the same way we calculate $P_{4,2}^{(10)}$, using C-K equation, we know

$$\mathbb{P}_{W,4,2}^{(10)} = \mathbb{P}_{W,4,A}^{(5)} \underbrace{\mathbb{P}_{W,A,2}^{(5)}}_{=0} + \underbrace{\mathbb{P}_{W,4,2}^{(5)} \mathbb{P}_{W,2,2}^{(5)}}_{=0} + \mathbb{P}_{W,4,3}^{(5)} \mathbb{P}_{W,3,2}^{(5)} + \underbrace{\mathbb{P}_{W,4,4}^{(5)} \mathbb{P}_{W,4,2}^{(5)}}_{=0}$$

in which

- $\mathbb{P}_{W,A,2}^{(5)} = 0$ because $\{W_n\}$ will never leave A.
- $\mathbb{P}_{W,4,2}^{(5)} = \mathbb{P}_{W,4,4}^{(5)} = 0$ because $\{W_n\}$ can never get from 4 to an even numbered state in odd numbers of steps.

Just need to find $\mathbb{P}_{W,4,3}^{(5)}$ and $\mathbb{P}_{W,3,2}^{(5)}$.

Example: Ehrenfest Model, 4 Balls (Cont'd)

$$P_W^{(2)} = \begin{array}{c} \\ A \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} A & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 3/8 & 0 & 1/8 \\ 3/8 & 0 & 5/8 & 0 \\ 0 & 3/4 & 0 & 1/4 \end{pmatrix} \end{array}, \quad P_W^{(3)} = \begin{array}{c} \\ A \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} A & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 11/16 & 0 & 5/16 & 0 \\ 3/8 & 15/32 & 0 & 5/32 \\ 3/8 & 0 & 5/8 & 0 \end{pmatrix} \end{array}$$

$$P_W^{(5)} = P_W^{(2)} \times P_W^{(3)} = \begin{array}{c} \\ A \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} A & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 0 & & \\ & 75/256 & & \\ & 0 & 25/64 & \end{pmatrix} \end{array}$$

So

$$\mathbb{P}_{W,4,2}^{(10)} = \mathbb{P}_{W,4,3}^{(5)} \, \mathbb{P}_{W,3,2}^{(5)} = \frac{25}{64} \times \frac{75}{256} = \frac{1875}{16384}.$$