First Step Analysis

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The Gambler's Ruin Problem

- A gambler starts with fortune $X_0 = i \in \{0, 1, \ldots, N\}$.
- Each play:

$$X_{n+1} = \begin{cases} X_n + 1, & \text{w.p. } p, \\ X_n - 1, & \text{w.p. } q = 1 - p, \end{cases} \qquad \text{independently over } n.$$

- The game stops when the gambler hits **bankruptcy** (0) or **target** (N).

$$P_{00} = P_{NN} = 1, \qquad P_{i,i+1} = p, \quad P_{i,i-1} = q, \quad i = 1, \ldots, N-1.$$

- Absorbing states: $\{0\}$ and $\{N\}$
- Transient states: $\{1, 2, \ldots, N-1\}$

Key Quantities

Define the absorption time

$$\tau := \inf\{n \geq 0 : X_n \in \{0, N\}\}.$$

Two standard questions:

1. **Hitting probability** (reach N before 0):

$$h_i := \mathbb{P}_i(\tau < \infty,\ X_\tau = N) = \mathbb{P}_i(\text{hit } N \text{ before } 0).$$

2. **Expected duration**:

$$m_i := \mathbb{E}_i[\tau].$$

Boundary conditions:

$$h_0 = 0,\ h_N = 1, \qquad m_0 = m_N = 0.$$

First-Step Analysis: Hitting Probability

Condition on the *first step* from state $i \in \{1, \dots, N-1\}$:

$$h_i = \mathbb{P}_i(\text{hit } N \text{ before } 0) = p\, h_{i+1} + q\, h_{i-1}.$$

So h_i solves the boundary value problem:

$$\begin{cases} h_i = p\, h_{i+1} + q\, h_{i-1}, \quad i = 1, \dots, N-1, \\ h_0 = 0,\ h_N = 1. \end{cases}$$

Closed Form for h_i

Case 1: fair walk $p = q = \frac{1}{2}$.

The recursion becomes $h_{i+1} - h_i = h_i - h_{i-1}$, so h_i is linear:

$$h_i = \frac{i}{N}.$$

Case 2: biased walk $p \neq q$.

Let $r := \frac{q}{p}$. Solving

$$h_i = p h_{i+1} + q h_{i-1}$$

gives

$$h_i = \frac{1 - r^i}{1 - r^N} = \frac{1 - (q/p)^i}{1 - (q/p)^N}.$$

Sanity checks:

- $h_0 = 0,\ h_N = 1.$
- If $p > q$, then $r < 1$ and h_i increases quickly with i.

Detailed Derivations

Let A be the event "hit N before 0". Then $h_i = \mathrm{P}(A \mid X_0 = i)$.

Conditioning on the outcome of the first game,

$$\begin{aligned} h_i &= \mathrm{P}(A \mid X_0 = i, \text{win 1st})\,\underbrace{\mathrm{P}(\text{win 1st})}_{=p} + \mathrm{P}(A \mid X_0 = i, \text{lose 1st})\,\underbrace{\mathrm{P}(\text{lose 1st})}_{=q} \\ &= \mathrm{P}(A \mid X_1 = i+1)\,p + \mathrm{P}(A \mid X_1 = i-1)\,q \qquad (\because \text{ Markov}) \\ &= \underbrace{h_{i+1}}_{\mathrm{P}(A|X_0=i+1)}\,p + \underbrace{h_{i-1}}_{\mathrm{P}(A|X_0=i-1)}\,q. \end{aligned}$$

Therefore,

$$h_i = ph_{i+1} + qh_{i-1}, \quad i = 1, 2, \ldots, N-1, \qquad h_0 = 0, \; h_N = 1.$$

Solving the equations $h_i = ph_{i+1} + qh_{i-1}$

$$
\begin{aligned}
(p+q)h_i &= ph_{i+1} + qh_{i-1} \qquad\qquad \text{since } p+q=1 \\
\Leftrightarrow \quad q(h_i - h_{i-1}) &= p(h_{i+1} - h_i) \\
\Leftrightarrow \quad h_{i+1} - h_i &= \left(\frac{q}{p}\right)(h_i - h_{i-1}).
\end{aligned}
$$

As $h_0 = 0$,

$$
\begin{aligned}
h_2 - h_1 &= \left(\frac{q}{p}\right)(h_1 - h_0) = \left(\frac{q}{p}\right)h_1, \\
h_3 - h_2 &= \left(\frac{q}{p}\right)(h_2 - h_1) = \left(\frac{q}{p}\right)^2 h_1, \\
&\vdots \\
h_i - h_{i-1} &= \left(\frac{q}{p}\right)^{i-1} h_1.
\end{aligned}
$$

Adding up gives

$$
h_i - h_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] h_1.
$$

Solving the equations $h_i = ph_{i+1} + qh_{i-1}$

$$(p+q)h_i = ph_{i+1} + qh_{i-1} \qquad \text{since } p+q=1$$

$$\Leftrightarrow \quad q(h_i - h_{i-1}) = p(h_{i+1} - h_i)$$

$$\Leftrightarrow \quad h_{i+1} - h_i = \left(\frac{q}{p}\right)(h_i - h_{i-1}).$$

As $h_0 = 0$,

$$h_2 - h_1 = \left(\frac{q}{p}\right)(h_1 - h_0) = \left(\frac{q}{p}\right)h_1,$$

$$h_3 - h_2 = \left(\frac{q}{p}\right)(h_2 - h_1) = \left(\frac{q}{p}\right)^2 h_1,$$

$$\vdots$$

$$h_i - h_{i-1} = \left(\frac{q}{p}\right)^{i-1} h_1.$$

Adding up gives

$$h_i - h_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] h_1.$$

Closed Form for h_i

From

$$h_i - h_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] h_1$$

we obtain

$$h_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)}\, h_1, & p \neq q, \\ i\, h_1, & p = q. \end{cases}$$

Using $h_N = 1$,

$$h_1 = \begin{cases} \dfrac{1-(q/p)}{1-(q/p)^N}, & p \neq q, \\ \dfrac{1}{N}, & p = q = \frac{1}{2}. \end{cases}$$

Therefore,

$$h_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N}, & p \neq \frac{1}{2}, \\ \dfrac{i}{N}, & p = \frac{1}{2}. \end{cases}$$

Closed Form

If $N = \infty$ (never quit),

$$\lim_{N \to \infty} h_i = \begin{cases} 1 - (q/p)^i, & p > \frac{1}{2}, \\ 0, & p \le \frac{1}{2}. \end{cases}$$

First-Step Analysis: Expected Time to Absorption

Again condition on the first step from $i \in \{1, \dots, N-1\}$. After one move, you have spent 1 step plus the remaining time:

$$m_i = 1 + p\, m_{i+1} + q\, m_{i-1}.$$

Boundary value problem:

$$\begin{cases} m_i = 1 + p\, m_{i+1} + q\, m_{i-1}, \quad i = 1, \dots, N-1, \\ m_0 = m_N = 0. \end{cases}$$

Closed Form for m_i

Case 1: fair walk $p = q = \frac{1}{2}$.

Solving $m_i = 1 + \frac{1}{2}m_{i+1} + \frac{1}{2}m_{i-1}$ with $m_0 = m_N = 0$ yields

$$m_i = i(N - i).$$

Case 2: biased walk $p \neq q$.

A convenient form (using h_i from before) is

$$m_i = \frac{i}{q-p} - \frac{N}{q-p}\, h_i = \frac{i}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^i}{1-(q/p)^N}.$$

Recursion for Expected Absorption Time

Recall that

$$m_i = \mathbb{E}_i[\tau]$$

satisfies, for $i = 1, \ldots, N-1$:

$$m_i = 1 + pm_{i+1} + qm_{i-1}$$

with boundary conditions:

$$m_0 = 0, \qquad m_N = 0.$$

Rearranging,

$$pm_{i+1} - m_i + qm_{i-1} = -1.$$

This is a **second-order linear difference equation**.

Step 1: Solve the Homogeneous Equation

First consider the homogeneous equation:

$$pm_{i+1} - m_i + qm_{i-1} = 0.$$

Try solutions of the form $m_i = r^i$.

Plug in:

$$pr^{i+1} - r^i + qr^{i-1} = 0$$

Divide by r^{i-1}:

$$pr^2 - r + q = 0.$$

This quadratic has roots:

$$r_1 = 1, \qquad r_2 = \frac{q}{p}.$$

So the homogeneous solution is:

$$m_i^{(h)} = A + B\left(\frac{q}{p}\right)^i.$$

Step 2: Find a Particular Solution

We now find a particular solution to:

$$pm_{i+1} - m_i + qm_{i-1} = -1.$$

Try a linear function:

$$m_i^{(p)} = Ci.$$

Substitute:

$$pC(i+1) - Ci + qC(i-1)$$

$$= C(pi + p - i + qi - q) = C((p+q-1)i + (p-q)).$$

Since $p + q = 1$, this simplifies to:

$$= C(p-q).$$

Set equal to -1:

$$C(p-q) = -1 \quad \Rightarrow \quad C = \frac{1}{q-p}.$$

So one particular solution is:

Step 3: General Solution

Combine homogeneous and particular parts:

$$m_i = A + B\left(\frac{q}{p}\right)^i + \frac{i}{q-p}.$$

Now apply boundary conditions.

From $m_0 = 0$:

$$A + B = 0 \quad \Rightarrow \quad A = -B.$$

From $m_N = 0$:

$$-B + B\left(\frac{q}{p}\right)^N + \frac{N}{q-p} = 0.$$

Solve for B:

$$B = \frac{N}{q-p} \cdot \frac{1}{1-(q/p)^N}.$$

Final Formula

Substitute constants back:

$$m_i = \frac{i}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^i}{1-(q/p)^N}.$$

Equivalently:

$$m_i = \frac{i}{q-p} - \frac{N}{q-p} h_i$$

where

$$h_i = \mathbb{P}_i(\text{hit } N \text{ before } 0).$$

Interpretation: Expected duration is a linear term minus a boundary correction.

Special Case: Fair Random Walk ($p = q = \frac{1}{2}$)

When $p = q$, the previous formula becomes singular.
Instead solve:

$$m_i = 1 + \tfrac{1}{2} m_{i+1} + \tfrac{1}{2} m_{i-1}.$$

Rewriting:

$$m_{i+1} - 2m_i + m_{i-1} = -2.$$

This has quadratic solution:

$$m_i = i(N - i).$$

So for a fair game:

$$\boxed{m_i = i(N - i)}.$$

Maximum expected duration occurs at $i = N/2$.

Expectation of the Absorbing State X_τ

Recall the absorption time

$$\tau = \inf\{n \geq 0 : X_n \in \{0, N\}\}.$$

At absorption, the random variable X_τ only takes two values:

$$X_\tau = \begin{cases} 0, & \text{with probability } 1 - h_i, \\ N, & \text{with probability } h_i, \end{cases}$$

where

$$h_i = \mathbb{P}_i(\text{hit } N \text{ before } 0).$$

Therefore,

$$\mathbb{E}_i[X_\tau] = 0 \cdot (1 - h_i) + N \cdot h_i = N h_i.$$

Special case: fair walk ($p = \frac{1}{2}$).
Since $h_i = \frac{i}{N}$,

$$\mathbb{E}_i[X_\tau] = i.$$

Use Recursive Relations of Markov Chains

Law of total expectation/variance

In many cases, we can use recursive relation to find $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$ without knowing the exact distribution of X_n.

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n])\end{aligned}$$

Example 1: Simple Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } q = 1 - p \end{cases}$$

So

$$\begin{aligned} \mathbb{E}[X_{n+1}|X_n] &= p(X_n + 1) + q(X_n - 1) = X_n + p - q \\ \mathrm{Var}[X_{n+1}|X_n] &= 4pq \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = \mathbb{E}[X_n] + p - q \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n]) \\ &= \mathbb{E}[4pq] + \mathrm{Var}(X_n + p - q) = 4pq + \mathrm{Var}(X_n) \end{aligned}$$

So

$$\mathbb{E}[X_n] = n(p - q) + \mathbb{E}[X_0], \qquad \mathrm{Var}(X_n) = 4npq + \mathrm{Var}(X_0)$$

Example 2: Ehrenfest Urn Model with M Balls

Recall that

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{M - X_n}{M} \\ X_n - 1 & \text{with probability } \frac{X_n}{M} \end{cases}$$

We have

$$\mathbb{E}[X_{n+1}|X_n] = (X_n + 1) \times \frac{M - X_n}{M} + (X_n - 1) \times \frac{X_n}{M} = 1 + \left(1 - \frac{2}{M}\right) X_n.$$

Thus

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = 1 + \left(1 - \frac{2}{M}\right) \mathbb{E}[X_n]$$

Subtracting $M/2$ from both sided of the equation above, we get

$$\mathbb{E}[X_{n+1}] - \frac{M}{2} = \left(1 - \frac{2}{M}\right) (\mathbb{E}[X_n] - \frac{M}{2})$$

Thus

$$\mathbb{E}[X_n] - \frac{M}{2} = \left(1 - \frac{2}{M}\right)^n (\mathbb{E}[X_0] - \frac{M}{2})$$

Random Walk w/ Reflective Boundary at 0

- State Space $= \{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p,\ P_{i,i-1} = 1 - p = q,$ for $i = 1, 2, 3 \ldots$
- Only one class, irreducible
- For $i < j$, define

$$\begin{aligned} N_{ij} &= \min\{m > 0 : X_m = j | X_0 = i\} \\ &= \text{first time to reach state } j \text{ when starting from state } i \end{aligned}$$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$
  By the Markov property, N_{01}, $N_{12}, \ldots, N_{n-1,n}$ are indep.
- Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i+1 \\ 1 + N^*_{i-1,i} + N^*_{i,i+1} & \text{if } X_1 = i-1 \end{cases} \qquad (1.1)$$

  Observe that $N^*_{i,i+1} \sim N_{i,i+1}$, and $N^*_{i,i+1}$ is indep of $N^*_{i-1,i}$.

Random Walk w/ Reflective Boundary at 0 (Cont'd)

Let $m_i = \mathbb{E}(N_{i,i+1})$. Taking expected value on Equation (1.1), we get

$$m_i = \mathbb{E}[N_{i,i+1}] = 1 + q\mathbb{E}[N^*_{i-1,i}] + q\mathbb{E}[N^*_{i,i+1}] = 1 + q(m_{i-1} + m_i)$$

Rearrange terms we get $pm_i = 1 + qm_{i-1}$ or

$$\begin{aligned} m_i &= \frac{1}{p} + \frac{q}{p} m_{i-1} \\ &= \frac{1}{p} + \frac{q}{p}(\frac{1}{p} + \frac{q}{p} m_{i-2}) \\ &= \frac{1}{p}\left[1 + \frac{q}{p} + (\frac{q}{p})^2 + \ldots + (\frac{q}{p})^{i-1}\right] + (\frac{q}{p})^i m_0 \end{aligned}$$

Since $N_{01} = 1$, which implies $m_0 = 1$.

$$m_i = \begin{cases} \frac{1-(q/p)^i}{p-q} + (\frac{q}{p})^i & \text{if } p \neq 0.5 \\ 2i + 1 & \text{if } p = 0.5 \end{cases}$$

Mean of $N_{0,n}$

Recall that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$

$$\begin{aligned}\mathbb{E}[N_{0n}] &= m_0 + m_1 + \ldots + m_{n-1} \\ &= \begin{cases} \frac{n}{p-q} - \frac{2pq}{(p-q)^2}[1 - (\frac{q}{p})^n] & \text{if } p \neq 0.5 \\ n^2 & \text{if } p = 0.5 \end{cases}\end{aligned}$$

When

$$\begin{array}{lll} p > 0.5 & \mathbb{E}[N_{0n}] \approx \frac{n}{p-q} - \frac{2pq}{(p-q)^2} & \text{linear in } n \\ p = 0.5 & \mathbb{E}[N_{0n}] = n^2 & \text{quadratic in } n \\ p < 0.5 & \mathbb{E}[N_{0n}] = O(\frac{2pq}{(p-q)^2}(\frac{q}{p})^n) & \text{exponential in } n \end{array}$$