Long-Term Behavior

Cong Ma

University of Chicago, Winter 2026

Outline

- Limiting distribution
- Stationary distribution
- Accessibility and communication
- Periodicity
- Recurrent and transient states
- Limit theorems for Markov chains

The Two-State Markov Chain

- **Setup:** State space $\{1, 2\}$ with transition matrix:

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \qquad 0 \leq p, q \leq 1$$

- **Degenerate Case ($p + q = 1$):**
  - Matrix has identical rows: $P = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}$
  - Stability: $P^n = P$ for all $n \geq 1$
  - Limiting distribution: $\pi = (1-p, p)$
- **General Case ($p + q \neq 1$):** Goal is to compute P^n explicitly.

Explicit Derivation of P^n

Focusing on entry P_{11}^n via the recursion $P^n = P^{n-1}P$:

$$\begin{aligned} P_{11}^n &= P_{11}^{n-1}(1-p) + P_{12}^{n-1}q \\ &= P_{11}^{n-1}(1-p) + (1 - P_{11}^{n-1})q \quad \text{(since rows sum to 1)} \\ &= q + (1-p-q)P_{11}^{n-1} \end{aligned}$$

Iterating the recursion/geometric sum yields the closed-form:

$$P_{11}^n = \frac{q}{p+q} + \frac{p}{p+q}(1-p-q)^n$$

The full matrix P^n is then:

$$P^n = \frac{1}{p+q}\begin{pmatrix} q + p(1-p-q)^n & p - p(1-p-q)^n \\ q - q(1-p-q)^n & p + q(1-p-q)^n \end{pmatrix}$$

Convergence and Limiting Behavior

- **Eigenvalues:** The behavior is governed by $\lambda_1 = 1$ and $\lambda_2 = 1 - p - q$.
- **Limit:** If $|1 - p - q| < 1$, then $\lim_{n\to\infty}(1 - p - q)^n = 0$:

$$\lim_{n\to\infty} P^n = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} \implies \pi = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$$

- **Key Takeaways:**
  - **Rate:** Convergence is exponential at rate $|1 - p - q|^n$.
  - **Ergodicity:** The chain loses memory of its initial state.
  - **Prototype:** This serves as the fundamental example for spectral analysis in finite-state chains.

Limiting Distribution

A probability distribution $\pi = [\pi_0, \pi_1, \pi_2, \ldots]$ is called the limiting distribution of a Markov chain X_n if for all $i, j \in \mathcal{X}$,

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_0 = i)$$

Matrix version

$$\text{i.e.,} \quad \lim_{n\to\infty} \mathbb{P}^{(n)} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proportion of Time in Each State

The limiting distribution gives the long-term probability that a Markov chain hits each state. It can also be interpreted as the long-term proportion of time that the chain visits each state.

To make this precise, let $X_0, X_1, \ldots$ be a Markov chain with transition matrix P and limiting distribution π. For state j, define indicator random variables

$$I_k = \begin{cases} 1, & \text{if } X_k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Limiting Distribution and Time Averages

For $k = 0, 1, \ldots$, the sum $\sum_{k=0}^{n-1} I_k$ is the number of times the chain visits state j in the first n steps (counting X_0 as the first step).

From initial state i, the long-term expected proportion of time that the chain visits j is

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{k=0}^{n-1} I_k \,\middle|\, X_0 = i\right) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}(I_k \mid X_0 = i) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}(X_k = j \mid X_0 = i) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P_{ij}^{(k)} \\
&= \lim_{n\to\infty} P_{ij}^{(n)} = \pi_j.
\end{aligned}$$

Back to Two-State Markov Chain

What happens if we assign the limiting distribution of a Markov chain to be the initial distribution of the chain?

Stationary distribution

It is interesting to consider what happens if we assign the limiting distribution of a Markov chain to be the initial distribution of the chain.

For the two-state chain, as in Example 3.1, the limiting distribution is

$$\pi = \left(\frac{q}{p+q},\ \frac{p}{p+q} \right).$$

Invariance of the limiting distribution

Let π be the initial distribution for such a chain. Then, the distribution of X_1 is

$$\pi P = \left(\frac{q}{p+q},\ \frac{p}{p+q}\right)\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$

Carrying out the multiplication,

$$\pi P = \left(\frac{q(1-p)+pq}{p+q},\ \frac{qp+p(1-q)}{p+q}\right).$$

Stationary distribution

Simplifying,

$$\pi P = \left(\frac{q}{p+q},\ \frac{p}{p+q} \right) = \pi.$$

That is, $\pi P = \pi$.

A probability vector π that satisfies

$$\pi P = \pi$$

plays a special role for Markov chains and is called a *stationary distribution*.

Limiting Distribution is a Stationary Distribution

The limiting distribution of a Markov chain is a stationary distribution of the Markov chain.

Proof By Chapman Kolmogorov Equation,

$$P_{ij}^{(n+1)} = \sum_{k \in \mathcal{X}} P_{ik}^{(n)} P_{kj}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
\pi_j = \lim_{n\to\infty} P_{ij}^{(n+1)} &= \lim_{n\to\infty} \sum_{k \in \mathcal{X}} P_{ik}^{(n)} P_{kj} \\
&=^* \sum_{k \in \mathcal{X}} \lim_{n\to\infty} P_{ik}^{(n)} P_{kj} \quad \text{(needs justification)} \\
&= \sum_{k \in \mathcal{X}} \pi_k P_{kj}
\end{aligned}$$

Thus the limiting distribution π_j's satisfies the equations $\pi_j = \sum_{k\in\mathcal{X}} \pi_k P_{kj}$ for all $j \in \mathcal{X}$ and is a stationary distribution.

Not All MCs Have a Stationary Distribution

For one-dimensional symmetric random walk, the transition probabilities are

$$P_{i,i+1} = P_{i,i-1} = 1/2$$

The stationary distribution $\{\pi_j\}$ would satisfy the equation:

$$\pi_j = \sum_{i \in \mathcal{X}} \pi_i P_{ij} = \frac{1}{2}\pi_{j-1} + \frac{1}{2}\pi_{j+1}.$$

Once π_0 and π_1 are determined, all π_j's can be determined from the equations as

$$\pi_j = \pi_0 + (\pi_1 - \pi_0)j, \quad \text{for all integer } j.$$

As $\pi_j \geq 0$ for all integer j, $\Rightarrow \pi_1 = \pi_0$. Thus

$$\pi_j = \pi_0 \quad \text{for all integer } j$$

Impossible to make $\sum_{j=-\infty}^{\infty} \pi_j = 1$.

Conclusion: 1-dim symmetric random walk does not have a stationary distribution.

Stationary Distribution May Not Be Unique

Consider a Markov chain with transition matrix $\mathbb{P}$ of the form

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \end{array} = \begin{pmatrix} \mathbb{P}_x & 0 \\ 0 & \mathbb{P}_y \end{pmatrix}$$

This Markov chain has 2 classes $\{0,1\}$ and $\{2, 3, 4\}$; both are recurrent. Note that this Markov chain can be reduced to two sub-Markov chains, one with state space $\{0,1\}$ and the other $\{2, 3, 4\}$. Their transition matrices are respectively $\mathbb{P}_x$ and $\mathbb{P}_y$.

Cont.

Say $\pi_x = (\pi_0, \pi_1)$ and $\pi_y = (\pi_2, \pi_3, \pi_4)$ be respectively the stationary distributions of the two sub-Markov chains, i.e.,

$$\pi_x \, \mathbb{P}_x = \pi_x, \quad \pi_y \, \mathbb{P}_y = \pi_y$$

Verify that $\pi = (c\pi_0, c\pi_1, (1-c)\pi_2, (1-c)\pi_3, (1-c)\pi_4)$ is a stationary distribution of $\{X_n\}$ for any c between 0 and 1.

Not All Markov Chains Have Limiting Distributions

Consider the simple random walk X_n on $\{0, 1, 2, 3, 4\}$ with absorbing boundary at 0 and 4. That is,

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability 0.5 if } 0 < X_n < 4 \\ X_n - 1 & \text{with probability 0.5 if } 0 < X_n < 4 \\ X_n & \text{if } X_n = 0 \text{ or } 4 \end{cases}$$

The transition matrix is hence

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Not All Markov Chains Have Limiting Distributions

The n-step transition matrix of the simple random walk X_n on $\{0, 1, 2, 3, 4\}$ with absorbing boundary at 0 and 4 can be shown by induction using the Chapman-Kolmogorov Equation to be

$$
\mathbb{P}^{(2n-1)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0.75 - 0.5^{n+1} & 0 & 0.5^{n} & 0 & 0.25 - 0.5^{n+1} \\
0.5 - 0.5^{n} & 0.5^{n} & 0 & 0.5^{n} & 0.5 - 0.5^{n} \\
0.25 - 0.5^{n+1} & 0 & 0.5^{n} & 0 & 0.75 - 0.5^{n+1} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

$$
\mathbb{P}^{(2n)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0.75 - 0.5^{n+1} & 0.5^{n+1} & 0 & 0.5^{n+1} & 0.25 - 0.5^{n+1} \\
0.5 - 0.5^{n+1} & 0 & 0.5^{n} & 0 & 0.5 - 0.5^{n+1} \\
0.25 - 0.5^{n+1} & 0.5^{n+1} & 0 & 0.5^{n+1} & 0.75 - 0.5^{n+1} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Not All Markov Chains Have Limiting Distributions

The limit of the n-step transition matrix as $n \to \infty$ is

$$\mathbb{P}^{(n)} \to \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.75 & 0 & 0 & 0 & 0.25 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0.25 & 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Though $\lim_{n\to\infty} P_{ij}^{(n)}$ exists but the limit depends on the initial state i, this Markov chain has no limiting distribution.

This Markov chain has two distinct absorbing states 0 and 4. Other transient states may be absorbed to either 0 or 4 with different probabilities depending how close those states are to 0 or 4.

When does a Markov chain have limiting distribution?

Accessibility

We say that state j is *accessible* from state i if

$$P_{ij}^{(n)} > 0 \quad \text{for some } n \geq 0.$$

That is, there is positive probability of reaching j from i in a finite number of steps.

States i and j *communicate* if i is accessible from j and j is accessible from i.

Accessibility is Transitive

Note that **accessibility is transitive**: for $i, j, k \in \mathcal{X}$,

$$\text{if } i \to j \text{ and } j \to k, \text{ then } i \to k.$$

Proof.

$$\begin{aligned} i \to j \quad &\Rightarrow \quad P_{ij}^{(m)} > 0 \text{ for some } m \\ j \to k \quad &\Rightarrow \quad P_{jk}^{(n)} > 0 \text{ for some } n \end{aligned}$$

By Chapman-Kolmogorov Equation:

$$P_{ik}^{(m+n)} = \sum_{l \in \mathcal{X}} P_{il}^{(m)} P_{lk}^{(n)} \geq P_{ij}^{(m)} P_{jk}^{(n)} > 0,$$

which shows $i \to k$.

Communication is an equivalence relation

Communication is an equivalence relation, which means that it satisfies the following three properties:

1. **(Reflexive)** Every state communicates with itself.
2. **(Symmetric)** If i communicates with j, then j communicates with i.
3. **(Transitive)** If i communicates with j and j communicates with k, then i communicates with k.

Communicating Class

Definition. Two states that communicate with each other are in the same **class**. A state that communicates with no other states itself is a class.

Fact. Two classes are either identical or disjoint.

Proof. If two classes A and B have one state i in common, then all states in A communicate with i and all states in B do too. Consequently, all states with A can communicate with states in B (through state i). Class A and Class B must be identical.

Example 1. Specify the classes of the following Markov chains.

$$\mathbb{P}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left(\begin{array}{cccc} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.9 & 0.1 \end{array} \right) \end{array} \qquad \mathbb{P}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left(\begin{array}{cccc} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

For $\mathbb{P}_1$, $1 \leftrightarrow 2 \rightarrow 3 \leftrightarrow 4$. Classes: $\{$1,2$\}$, $\{$3,4$\}$.

For $\mathbb{P}_2$, $\begin{array}{ccc} 1 & \longleftrightarrow & 2 \\ & \nwarrow & \uparrow \\ 4 & \leftarrow & 3 \\ \circlearrowleft & & \circlearrowleft \end{array}$. Classes: $\{$1,2$\}$, $\{$3$\}$, $\{$4$\}$.

Example 2. How many classes does the Ehrenfest diffusion model with K balls have?

All states communicate. Only one class.

Irreducibility

Definition 1.1 (Irreducibility)

A Markov chain is called *irreducible* if it has exactly one communication class. That is, all states communicate with each other.

Periodicity

A state of a Markov chain is said to have **period** d if

$$P_{ii}^{(n)} = 0, \quad \text{whenever } n \text{ is not a multiple of } d$$

In other words, d is the *greatest common divisor* of all the n's such that

$$P_{ii}^{(n)} > 0$$

We say a state is **aperiodic** if $d = 1$, and **periodic** if $d > 1$.

Fact: Periodicity is a class property.
That is, all states in the same class have the same period.

For a proof, see Problem 2&3 on p.77 of Karlin & Taylor (1975).

Examples (Periodicity)

- All states in the Ehrenfest diffusion model are of period $d = 2$ since it's impossible to move back to the initial state in odd number of steps.
- 1-D (2-D) Simple random walk on all integers (grids on a 2-d plane) are of period $d = 2$

Example (Periodicity)

Specify the classes of a Markov chain with the following transition matrix, and find the periodicity for each state.

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.1 & 0.9 \\
0 & 0 & 0 & 0 & 0 & 0.7 & 0.3
\end{pmatrix}
\end{array}
$$

Classes: $\{1,2,3,4,5\}$, $\{6,7\}$.

Example (Periodicity)

Specify the classes of a Markov chain with the following transition matrix, and find the periodicity for each state.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.1 & 0.9 \\
0 & 0 & 0 & 0 & 0 & 0.7 & 0.3
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{ccccc}
5 & \rightarrow & 1 & \rightarrow & 2 \\
\uparrow & \swarrow & \uparrow & \swarrow & \\
4 & \rightarrow & 3 & & \\
 & & \downarrow & & \\
7 & \leftrightarrow & 6 & &
\end{array}
$$

Classes: {1,2,3,4,5}, {6,7}.

Period is $d = 1$ for state 6 and 7.

Example (Periodicity)

Specify the classes of a Markov chain with the following transition matrix, and find the periodicity for each state.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.1 & 0.9 \\
0 & 0 & 0 & 0 & 0 & 0.7 & 0.3
\end{pmatrix}
\end{array}
$$

$$
\begin{array}{ccccc}
5 & \rightarrow & 1 & \rightarrow & 2 \\
\uparrow & \swarrow & \uparrow & \swarrow & \\
4 & \rightarrow & 3 & & \\
 & & \downarrow & & \\
7 & \leftrightarrow & 6 & &
\end{array}
$$

Classes: {1,2,3,4,5}, {6,7}.
Period is $d = 1$ for state 6 and 7.
Period is $d = 3$ for state 1,2,3,4,5 since
$\{1\} \rightarrow \{2, 4\} \rightarrow \{3, 5\} \rightarrow \{1\}$.

Periodic Markov Chains Have No Limiting Distributions

For example, in the Ehrenfest diffusion model with 4 balls, it can be shown by induction that the $(2n-1)$-step transition matrix is

$$\mathbb{P}^{(2n-1)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1/2+1/2^{2n-1} & 0 & 1/2-1/2^{2n-1} & 0 \\ 1/8+1/2^{2n+1} & 0 & 3/4 & 0 & 1/8-1/2^{2n+1} \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/8-1/2^{2n+1} & 0 & 3/4 & 0 & 1/8+1/2^{2n+1} \\ 0 & 1/2-1/2^{2n-1} & 0 & 1/2+1/2^{2n-1} & 0 \end{pmatrix} \end{array}$$

$$\to \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{pmatrix} \end{array} \quad \text{as } n \to \infty.$$

Periodic Markov Chains Have No Limiting Distributions

and the $2n$-step transition matrix is

$$\mathbb{P}^{(2n)} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/8+1/2^{2n+1} & 0 & 3/4 & 0 & 1/8-1/2^{2n+1} \\ 0 & 1/2+1/2^{2n+1} & 0 & 1/2-1/2^{2n+1} & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 1/2-1/2^{2n+1} & 0 & 1/2+1/2^{2n+1} & 0 \\ 1/8-1/2^{2n+1} & 0 & 3/4 & 0 & 1/8+1/2^{2n+1} \end{pmatrix} \end{array}$$

$$\to \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \end{pmatrix} \end{array} \quad \text{as } n \to \infty.$$

Periodic Markov Chains Have No Limiting Distributions

In general for Ehrenfest diffusion model with N balls, as $n \to \infty$,

$$P_{ij}^{(2n)} \to \begin{cases} 2\binom{N}{j}(\frac{1}{2})^N & \text{if } i+j \text{ is even} \\ 0 & \text{if } i+j \text{ is odd} \end{cases}$$

$$P_{ij}^{(2n+1)} \to \begin{cases} 0 & \text{if } i+j \text{ is even} \\ 2\binom{N}{j}(\frac{1}{2})^N & \text{if } i+j \text{ is odd} \end{cases}$$

$\lim_{n\to\infty} P_{ij}^{(n)}$ doesn't exist for all $i, j \in \mathcal{X}$

First passage time

Given a Markov chain $X_0, X_1, \ldots$, let

$$T_j = \min\{n > 0 : X_n = j\}$$

be the *first passage time* to state j. If $X_n \neq j$ for all $n > 0$, set $T_j = \infty$.

Let

$$f_j = \mathbb{P}(T_j < \infty \mid X_0 = j)$$

be the probability that the chain started in j eventually returns to j.

Recurrent and transient states

Definition 1.2 (Recurrent and transient states)

State j is said to be *recurrent* if the Markov chain started in j eventually revisits j. That is,

$$f_j = 1.$$

State j is said to be *transient* if there is positive probability that the Markov chain started in j never returns to j. That is,

$$f_j < 1.$$

Recurrence and Transience: Another Characterization

Theorem 1.3 (Recurrence and transience)

1. *State j is recurrent if and only if*

$$\sum_{n=0}^{\infty} P_{jj}^{(n)} = \infty.$$

2. *State j is transient if and only if*

$$\sum_{n=0}^{\infty} P_{jj}^{(n)} < \infty.$$

Proof

Suppose that $X_0 = i$, and consider the random variable
$N(i) = \sum_{n=1}^{\infty} 1\{X_n = i\}$
We will use two ways to calculate the expectation of $N(i)$. First, by definition we have

$$\begin{aligned}\mathbb{E}[N(i)] &= \mathbb{E}[\sum_{n=1}^{\infty} 1\{X_n = i\}] = \sum_{n=1}^{\infty} \mathbb{E}[1\{X_n = i\}] \\ &= \sum_{n=1}^{\infty} \mathbb{P}\{X_n = i\}] = \sum_{n=1}^{\infty} P_{ii}^{(n)}\end{aligned}$$

In addition, we have

$$\mathbb{E}[N(i)] = \sum_{k=0}^{\infty} \mathbb{P}(N(i) \geq k) = \sum_{k=0}^{\infty} f_i^k$$

Recurrence and transience are class properties

Theorem 1.4 (Class property of recurrence and transience)

The states of a communication class are either all recurrent or all transient.

Proof

$$i \to j \quad \Rightarrow \quad P_{ij}^{(k)} > 0 \text{ for some } k$$

$$j \to i \quad \Rightarrow \quad P_{ji}^{(l)} > 0 \text{ for some } l$$

By Chapman-Kolmogorov Equation:

$$P_{jj}^{(l+n+k)} \geq P_{ji}^{(l)} P_{ii}^{(n)} P_{ij}^{(k)}, \text{ for all } k = 0, 1, 2, \ldots$$

Thus

$$\sum_{n=1}^{\infty} P_{jj}^{(n)} \geq \sum_{n=1}^{\infty} P_{jj}^{(l+n+k)} \geq \underbrace{P_{ji}^{(l)}}_{>0} \underbrace{\sum_{n=1}^{\infty} P_{ii}^{(n)}}_{=\infty} \underbrace{P_{ij}^{(k)}}_{>0} = \infty$$

Consequences for accessibility

Assume that state j is recurrent and accessible from state i. Then, for the chain started in i:

- there is positive probability of hitting j,
- starting from j, the expected number of visits to j is infinite.

It follows that the expected number of visits to j for the chain started in i is also infinite, and thus

$$\sum_{n=0}^{\infty} P_{ij}^{(n)} = \infty.$$

Transient states and limiting probabilities

Assume that state j is transient and accessible from state i. By a similar argument, the expected number of visits to j for the chain started in i is finite, and hence

$$\sum_{n=0}^{\infty} P_{ij}^{(n)} < \infty.$$

From this it follows that

$$\lim_{n\to\infty} P_{ij}^{(n)} = 0. \tag{3.5}$$

Interpretation

The long-term probability that a Markov chain eventually hits a transient state is zero.

Finite irreducible Markov chains

Corollary 1.5

For a finite irreducible Markov chain, all states are recurrent.

Proof

First based on the previous corollary, we know either all the states are transient, or all the states are recurrent. Suppose that all the states are transient. Then for all $i \in \mathcal{X}$, we have

$$\lim_{n\to\infty} P_{0i}^{(n)} = 0.$$

Since we have a finite state space, we obtain

$$\lim_{n\to\infty} \sum_{i\in\mathcal{X}} P_{0i}^{(n)} = \sum_{i\in\mathcal{X}} \lim P_{0i}^{(n)} = 0.$$

However, the left hand is equal to 1. This marks a contradiction. Hence the chain cannot be transient.

Example: One-Dimensional Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob. } p \\ X_n - 1 & \text{with prob. } 1-p \end{cases}$$

- State space $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
- All states communicate

$$\cdots \longleftrightarrow -2 \longleftrightarrow -1 \longleftrightarrow 0 \longleftrightarrow 1 \longleftrightarrow 2 \longleftrightarrow \cdots$$

Only one class $\Rightarrow$ Irreducible

$\Rightarrow$ States are all transient or all recurrent.

It suffices to check whether 0 is recurrent or transient, i.e., whether

$$\sum_{n=1}^{\infty} P_{00}^{(n)} = \infty \text{ or } < \infty$$

Example: One-Dimensional Random Walk (Cont'd)

$$P_{00}^{(2n+1)} = 0 \quad \text{(Why?)}$$

$$\begin{aligned} P_{00}^{(2n)} &= \binom{2n}{n} p^n (1-p)^n \\ &= \frac{(2n)!}{n!\,n!} p^n (1-p)^n \\ &\approx \frac{(2n)^{2n+0.5} e^{-2n}\sqrt{2\pi}}{(n^{n+0.5}e^{-n}\sqrt{2\pi})^2} p^n (1-p)^n \\ &= \frac{1}{\sqrt{\pi n}} [4p(1-p)]^n \end{aligned}$$

Stirling's Formula: $n! \approx n^{n+0.5} e^{-n} \sqrt{2\pi}$

Thus $$\sum_{n=1}^{\infty} P_{ii}^{2n} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} [4p(1-p)]^n \begin{cases} < \infty & \text{if } p \neq 1/2 \\ = \infty & \text{if } p = 1/2 \end{cases}$$

Conclusion: One-dimensional random walk is recurrent if $p = 1/2$, and transient otherwise.

Example: Two-Dimensional Symmetric Random Walk

Irreducible. Just check if 0 is recurrent.

$$
\begin{aligned}
P_{00}^{(2n)} &= \sum_{i=0}^{n} \frac{(2n)!}{i!i!(n-i)!(n-i)!} \left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n} \underbrace{\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i}}_{=\binom{2n}{n}} \left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n}^2 \left(\frac{1}{4}\right)^{2n} \approx \frac{1}{\pi n} \quad \text{by Stirling's Formula}
\end{aligned}
$$

Thus $\sum_{n=1}^{\infty} P_{00}^{(2n)} = \infty$.Two-dimensional symmetric random walk is **recurrent.**

Example: d-Dimensional Symmetric Random Walk

In general, for a d-dimensional symmetric random walk, it can be shown that

$$P_{00}^{(2n)} \approx (1/2)^{d-1} \left(\frac{d}{n\pi}\right)^{d/2}$$

Thus

$$\sum_{n=1}^{\infty} P_{00}^{(2n)} \begin{cases} = \infty & \text{for } d = 1 \text{ or } 2 \\ < \infty & \text{for } d \geq 3 \end{cases}.$$

"*A drunken man will find his way home.*
A drunken bird might be lost forever."

Positive Recurrence and Null Recurrence

Recall the first passage time of a state i

$$T_i = \min\{n > 0 : X_n = i \mid X_0 = i\}$$

We say a state i is

- **positive recurrent** if i is recurrent and $\mathbb{E}[T_i] < \infty$.
- **null recurrent** if i is recurrent but $\mathbb{E}[T_i] = \infty$.

Positive and Null Recurrence

Lemma 1.6 (Class property of positive and null recurrence)

All the states in a recurrent communication class are either positive recurrent or null recurrent.

The Fundamental Limit Theorem of Markov Chain II

For an **irreducible** Markov chain, it is **positive recurrent** if and only if there exists a stationary distribution, i.e., a solution to the set of equations:

$$\pi_i \geq 0, \quad \sum_{i \in \mathcal{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathcal{X}} \pi_i P_{ij}$$

Moreover, if a solution exists then it is unique, and is given by

$$\pi_j = \frac{1}{\mathbb{E}[T_j]} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)}.$$

Stationary distribution can be interpreted as the **long run proportion of time that the Markov chain is in state** j.

Heuristic proof

Step 1: Connecting long run proportion of time to inverse expected return time, i.e., we aim to show that for any state j, we have

$$\mathbb{P}_j \left[\lim_{n \to \infty} \frac{\sum_{i=1}^{n} 1\{X_i = j\}}{n} = \frac{1}{\mathbb{E}_j[T_j]} \right] = 1$$

If j is transient, both are 0.
If j is recurrent, see the next slide

When j is recurrent

Consider a Markov chain started from state j. Let S_k be the time till the k-th visit to state j. Then

$$S_k = T_{jj}(0) + T_{jj}(1) + \ldots + T_{jj}(k-1)$$

Here

- $T_{jj}(m) =$ the time between the mth and $(m+1)$st visit to state j.

Observe that $T_{jj}(0)$, $T_{jj}(1), \ldots T_{jj}(k-1)$ are i.i.d. and have the same distribution as T_i.
For k large, the Strong Law of Large Numbers tells us

$$\frac{1}{k}[T_{jj}(0) + T_{jj}(1) + \cdots + T_{jj}(k-1)] \to \mathbb{E}_j(T_j) \quad \text{almost surely}$$

i.e., the chain visits state j about k times in $k\,\mathbb{E}(T_j)$ steps.

Heuristic proof

Step 2: Connecting long run proportion of time to stationary probability
Consider a Markov chain starting from the stationary distribution. Then in n steps, we expect about $n\pi(j)$ visits to the state j. Hence

$$\pi_j$$

is roughly the proportion of time we see j.

Fundamental Limit Theorem for Ergodic Markov Chains

Theorem 1.7 (Fundamental Limit Theorem for Ergodic Markov Chains)

Let $X_0, X_1, \ldots$ be an ergodic Markov chain. There exists a unique, positive, stationary distribution π, which is the limiting distribution of the chain.

That is,

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}, \qquad \text{for all } i, j.$$

Example 1: One-Dimensional Random Walk

We have shown that 1-dim symmetric random walk has no stationary distribution.

- Conclusion from 2nd limit theorem: 1-dim symmetric random walk is null recurrent, i.e.

$$\mathbb{E}[T_i] = \infty \quad \text{for all state } i$$

In fact, in Lecture 3 we have shown that

$$P_{ii}^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2}\left(\frac{1}{2}\right)^n \approx \sqrt{\frac{2}{\pi n}} & \text{if } n \text{ is even} \end{cases}$$

Thus we see $\lim_{n\to\infty} P_{ii}^{(n)} = 1/\,\mathbb{E}[T_i]$.

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$\begin{aligned} P_{i,i+1} &= p \quad \text{for all } i = 0, 1, 2, \ldots \\ P_{i,i-1} &= 1-p \quad \text{for all } i = 1, 2, \ldots \\ p_{00} &= 1-p \end{aligned}$$

Try to solve $\pi_j = \sum_{i \in \mathcal{X}} \pi_i P_{ij}$

$$\begin{aligned} \pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} = (1-p)(\pi_0 + \pi_1) & \Rightarrow \pi_1 &= \tfrac{p}{1-p}\pi_0 \\ \pi_1 &= \pi_0 P_{01} + \pi_2 P_{21} = p\pi_0 + (1-p)\pi_2 & \Rightarrow \pi_2 &= \left(\tfrac{p}{1-p}\right)^2 \pi_0 \\ \pi_2 &= \pi_0 P_{12} + \pi_3 P_{32} = p\pi_1 + (1-p)\pi_3 & \Rightarrow \pi_3 &= \left(\tfrac{p}{1-p}\right)^3 \pi_0 \\ &\vdots \\ \pi_j &= p\pi_{j-1} + (1-p)\pi_{j+1} & \Rightarrow \pi_{j+1} &= \left(\tfrac{p}{1-p}\right)^{j+1} \pi_0 \end{aligned}$$

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \left(\frac{p}{1-p} \right)^i = \begin{cases} \pi_0 \left(\frac{1-p}{1-2p} \right) & \text{if } p < 1/2 \\ \infty & \text{if } p \geq 1/2 \end{cases}$$

Conclusion: The process is positive recurrent iff $p < 1/2$, in which case

$$\pi_i = \frac{1-2p}{1-p} \left(\frac{p}{1-p} \right)^i, \quad i = 0, 1, 2, \ldots$$

Example 3: Ehrenfest Diffusion Model with N Balls

$$P_{ij} = \begin{cases} \dfrac{i}{N} & \text{if } j = i-1 \\ \dfrac{N-i}{N} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\pi_0 &= \pi_1 P_{10} = \tfrac{1}{N}\pi_1 \Rightarrow \pi_1 = N\pi_0 = \tbinom{N}{1}\pi_0 \\
\pi_1 &= \pi_0 P_{01} + \pi_2 P_{21} = \pi_0 + \tfrac{2}{N}\pi_2 \Rightarrow \pi_2 = \tfrac{N(N-1)}{2}\pi_0 = \tbinom{N}{2}\pi_0 \\
\pi_2 &= \pi_1 P_{12} + \pi_3 P_{32} = \tfrac{N-1}{N}\pi_1 + \tfrac{3}{N}\pi_3 \Rightarrow \pi_3 = \tfrac{N(N-1)(N-2)}{6}\pi_0 = \tbinom{N}{3}\pi_0 \\
&\vdots \qquad \vdots
\end{aligned}$$

In general, you'll get $\pi_i = \binom{N}{i}\pi_0$.
As $1 = \sum_{i=0}^{N} \pi_i = \pi_0 \sum_{i=0}^{N} \binom{N}{i}$ and $\sum_{i=0}^{N} \binom{N}{i} = 2^N$, we have

$$\pi_i = \binom{N}{i}\left(\frac{1}{2}\right)^N \quad \text{for } i = 0, 1, 2, \ldots, N.$$

Though the limiting distribution $\lim_{n\to\infty} P_{ij}^{(n)}$ does not exist, we can show that

$$\lim_{n\to\infty} P_{ij}^{(2n)} = 2\binom{N}{j}(\frac{1}{2})^N, \quad \lim_{n\to\infty} P_{ij}^{(2n+1)} = 0 \quad \text{if } i+j \text{ is even}$$

$$\lim_{n\to\infty} P_{ij}^{(2n)} = 0, \quad \lim_{n\to\infty} P_{ij}^{(2n+1)} = 2\binom{N}{j}(\frac{1}{2})^N \quad \text{if } i+j \text{ is odd}$$

From the above, one can verify that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^{(k)} = \binom{N}{j}(\frac{1}{2})^N = \pi_j$.

Exercise 4.50 on p.284

A Markov chain has transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

Communicating classes:

$$\begin{array}{ccc} \{1,2\} & \{3,4\} & \{5,6\} \\ \uparrow & \uparrow & \uparrow \\ \text{transient} & \text{recurrent} & \text{recurrent} \end{array}$$

Find $\lim_{n\to\infty} P^{(n)}$.

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is transient, hence,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is NOT accessible from i

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

The two classes {3,4} and {5,6} do not communicate and hence the transition probabilities in between are all 0.

Exercise 4.50 on p.284 (Cont'd)

Recall we have shown that the limiting distribution of a two-state Markov chain with the transition matrix $\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ is $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$. As the Markov chain restricted to the class $\{3,4\}$ is also a Markov chain with the transition matrix

$$\begin{array}{c} \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \end{array}.$$

Hence,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

For the same reason,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{array}$$