Martingales

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Stochastic Processes and Information

A **discrete-time stochastic process** is a sequence:

$$X_0, X_1, X_2, \ldots$$

Let $\mathcal{F}_n$ be the information available up to time n.

- $(\mathcal{F}_n)$ is called a **filtration**
- Information increases over time

Martingales

Definition 1.1

A process (X_n) is a **martingale** if:

1. $E|X_n| < \infty$
2. $$E[X_{n+1} \mid \mathcal{F}_n] = X_n$$

Martingales = Fair Games

The martingale condition means:

- Given all current information
- The expected future value equals the present value

Consequences:

- No systematic profit
- No predictable drift

Martingales model fair games and fair systems.

A Key Property

If (X_n) is a martingale, then for every fixed n:

$$E[X_n] = E[X_0]$$

Interpretation:

- Expected value is conserved over time
- Martingales behave like conservation laws

Example: Partial Sums of Independent Variables

Let

$$X_1, X_2, \ldots$$

be independent random variables with common mean

$$E[X_i] = \mu.$$

Define the partial sums:

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n.$$

Interpretation:

- S_n represents the accumulated total after n steps
- Examples: random walk, cumulative gains, repeated experiments

Centered Partial Sums Form a Martingale

Define the centered process:

$$M_n = S_n - n\mu.$$

We claim that (M_n) is a martingale.

Compute the conditional expectation given past observations:

$$E[M_{n+1} \mid X_1, \ldots, X_n] = E[S_{n+1} - (n+1)\mu \mid X_1, \ldots, X_n]$$

Verification of the Martingale Property

Since

$$S_{n+1} = S_n + X_{n+1}$$

and X_{n+1} is independent of the past with mean μ:

$$E[S_{n+1} \mid X_1, \ldots, X_n] = S_n + \mu$$

Therefore:

$$E[M_{n+1} \mid X_1, \ldots, X_n] = (S_n + \mu) - (n+1)\mu = S_n - n\mu = M_n$$

Hence (M_n) is a martingale.

Example: Doubling Betting Strategy

Let $X_1, X_2, \ldots$ be independent with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Interpretation: win \$1 for heads, lose \$1 for tails.

Doubling strategy:

- Bet \$1 initially
- After each loss, double the next bet: $1, 2, 4, 8, \ldots$
- Stop at the first win

Let W_n be total winnings after n flips, with $W_0 = 0$.

Why This Is Still a Martingale

After n consecutive losses:

$$W_n = -(2^n - 1)$$

Next flip:

- Win with prob. $\frac{1}{2}$: $\quad W_{n+1} = 1$
- Lose with prob. $\frac{1}{2}$: $\quad W_{n+1} = -(2^{n+1} - 1)$

Expected value:

$$E[W_{n+1} \mid \text{past}] = \frac{1}{2}(1) + \frac{1}{2}\big(- (2^{n+1} - 1)\big) = W_n$$

Conclusion: (W_n) is a martingale.

Martingales Preserve Expectation

If (X_n) is a martingale, then for every fixed time n:

$$E[X_n] = E[X_0].$$

Interpretation:

- No drift in expectation
- Models a fair system
- Expected value is conserved over time

The Big Question

We know:

$$E[X_n] = E[X_0] \quad \text{for fixed } n.$$

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Question

If T is a stopping time, is it still true that

$$E[X_T] = E[X_0] \ ?$$

Stopping Times: Definition

Definition 1.2

A random time T is a stopping time if for all n:

$$\{T = n\} \in \mathcal{F}_n \quad \text{(equivalently } \{T \le n\} \in \mathcal{F}_n\text{)}.$$

Stopping Times: Interpretation

- Whether we stop at time n is decided using past information only
- No access to future outcomes

Stopping rules must be **causal**.

Stopping Times: Examples

- First time a random walk hits zero
- First time gambler reaches a target fortune
- First time a population becomes extinct

Example: Gambler's Ruin

A gambler starts with i dollars.
Each round:

- Win \$1 with probability $1/2$
- Lose \$1 with probability $1/2$

Let X_n be wealth after n games.

Game ends when reaching 0 or N dollars.

Martingale Property

Conditioning on $\mathcal{F}_n$:

$$E[X_{n+1} \mid \mathcal{F}_n] = X_n + \frac{1}{2} - \frac{1}{2} = X_n$$

Therefore:

(X_n) is a martingale

Stopping Time for Gambler's Ruin

Define:

$$T = \inf\{n : X_n = 0 \text{ or } X_n = N\}$$

This is a stopping time because:

- We only use information available at time n
- No future values are inspected

Optional Stopping Theorem

Theorem 1.3

Let (X_n) be a martingale and T a stopping time.

Then

$$E[X_T] = E[X_0]$$

provided that at least one holds:

1. ***Bounded process:*** $|X_n| \leq C$
2. ***Bounded stopping time:*** $T \leq K$
3. ***Finite expected time:*** $E[T] < \infty$ *and*

$$|X_{n+1} - X_n| \leq C$$

What Does Optional Stopping Mean?

If the stopping rule is **reasonable**, then:

$$E[X_T] = E[X_0]$$

Meaning:

- You may choose when to stop
- Your rule may be complicated
- But expectation cannot be changed

You cannot beat a fair system by timing alone.

Applying to Gambler's Ruin

At stopping time:

$$X_T = \begin{cases} 0 & \text{with probability } P(\text{ruin}) \\ N & \text{with probability } P(\text{win}) \end{cases}$$

So:

$$E[X_T] = N \cdot P(\text{win})$$

Since $E[X_T] = i$:

$$P(\text{win}) = \frac{i}{N}$$

Why Conditions Matter

Consider symmetric random walk:

$$S_n = \sum_{k=1}^{n} Y_k, \quad P(Y_k = \pm 1) = \frac{1}{2}$$

Then:

$$E[S_n] = 0$$

and (S_n) is a martingale.

A Dangerous Stopping Time

Define:

$$T = \inf\{n : S_n = 1\}$$

- T is almost surely finite
- But $E[T] = \infty$

At stopping:

$$S_T = 1$$

So:

$$E[S_T] = 1 \neq 0$$

Optional stopping fails.

Lesson

Optional stopping can fail if:

- Stopping time is too large
- Integrability conditions fail

Assumptions are not technical details they are essential.