Poisson Processes

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Exponential Distribution

Let X follow exponential distribution with rate λ: $X \sim Exp(\lambda)$.

- Density: $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- CDF: $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$
- $\mathbb{E}(X) = 1/\lambda$, $\mathrm{V}ar(X) = 1/\lambda^2$
- If $X_1, \ldots, X_n$ are i.i.d $Exp(\lambda)$, then $S_n = X_1 + \cdots + X_n \sim Gamma(n, \lambda)$, with density

$$f_{S_n}(x) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

The Exponential Distribution is Memoryless ($\star\star\star\star\star$)

Lemma: for all $s, t \geq 0$

$$\mathrm{P}(X > t + s \mid X > t) = \mathrm{P}(X > s)$$

Proof.

$$\begin{aligned}
\mathrm{P}(X > t + s | X > t) &= \frac{\mathrm{P}(X > t + s \text{ and } X > t)}{\mathrm{P}(X > t)} \\
&= \frac{\mathrm{P}(X > t + s)}{\mathrm{P}(X > t)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathrm{P}(X > s)
\end{aligned}$$

Implication. If the lifetime of batteries has an Exponential distribution, then *a used battery is as good as a new one*, as long as it's not dead!

Another Important Property of the Exponential

If $X_1, \ldots, X_n$ are independent, $X_i, \sim Exp(\lambda_i)$ for $i = 1, \ldots, n$ then

(i) $\min(X_1, \ldots, X_n) \sim Exp(\lambda_1 + \cdots + \lambda_n)$, and

(ii) $\mathrm{P}(X_j = \min(X_1, \ldots, X_n)) = \dfrac{\lambda_j}{\lambda_1 + \cdots + \lambda_n}$

Proof of (i)

$$\begin{aligned} &\mathrm{P}(\min(X_1, \ldots, X_n) > t) = \mathrm{P}(X_1 > t, \ldots, X_n > t) \\ &= \mathrm{P}(X_1 > t) \ldots \mathrm{P}(X_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} \\ &= e^{-(\lambda_1 + \cdots + \lambda_n)t}. \end{aligned}$$

Proof of (ii)

$$
\begin{aligned}
& \mathrm{P}(X_j = \min(X_1, \ldots, X_n)) \\
&= \mathrm{P}(X_j < X_i \text{ for } i = 1, \ldots, n, i \neq j) \\
&= \int_0^\infty \mathrm{P}(X_j < X_i \text{ for } i \neq j | X_j = t) \lambda_j e^{-\lambda_j t} dt \\
&= \int_0^\infty \mathrm{P}(t < X_i \text{ for } i \neq j) \lambda_j e^{-\lambda_j t} dt \\
&= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} \mathrm{P}(X_i > t) dt \\
&= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} e^{-\lambda_i t} dt \\
&= \lambda_j \int_0^\infty e^{-(\lambda_1 + \cdots + \lambda_n)t} dt \\
&= \frac{\lambda_j}{\lambda_1 + \cdots + \lambda_n}
\end{aligned}
$$

Post Office

- A post office has two clerks.
- Service times for clerk $i \sim Exp(\lambda_i)$, $i = 1$, 2
- When you arrive, both clerks are busy but no one else waiting. You will enter service when either clerk becomes free.
- Find $\mathbb{E}[T]$, where $T =$ the amount of time you spend in the post office.

Solution. Let $R_i =$ remaining service time of the customer with clerk i, $i = 1$, 2.

- Note R_i's are indep. $\sim Exp(\lambda_i)$, $i = 1$, 2 by the memoryless property
- Observe $T = \min(R_1, R_2) + S$ where S is your service time
- Using the property of exponential distributions,

$$\min(R_1, R_2) \sim Exp(\lambda_1 + \lambda_2) \quad \Rightarrow \quad \mathbb{E}[\min(R_1, R_2)] = \frac{1}{\lambda_1 + \lambda_2}$$

Post Office (Cont'd)

As for your service time S, observe that

$$S \sim \begin{cases} Exp(\lambda_1) & \text{if } R_1 < R_2 \\ Exp(\lambda_2) & \text{if } R_2 < R_1 \end{cases} \Rightarrow \begin{array}{l} \mathbb{E}[S|R_1 < R_2] = 1/\lambda_1 \\ \mathbb{E}[S|R_2 < R_1] = 1/\lambda_2 \end{array}$$

Recall that $\mathrm{P}(R_1 < R_2) = \lambda_1/(\lambda_1 + \lambda_2)$ So

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[S|R_1 < R_2]\mathrm{P}(R_1 < R_2) + \mathbb{E}[S|R_2 < R_1]\mathrm{P}(R_2 < R_1) \\ &= \frac{1}{\lambda_1} \times \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2} \times \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{2}{\lambda_1 + \lambda_2} \end{aligned}$$

Hence the expected amount of time you spend in the post office is

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}[\min(R_1, R_2)] + \mathbb{E}[S] \\ &= \frac{1}{\lambda_1 + \lambda_2} + \frac{2}{\lambda_1 + \lambda_2} = \frac{3}{\lambda_1 + \lambda_2}. \end{aligned}$$

5.3.1. Counting Processes

A counting process $\{N(t)\}$ is a cumulative count of number of events happened up to time t.

Definition.

A stochastic processes $\{N(t), t \geq 0\}$ is a *counting process* satisfying

(i) $N(t) = 0, 1, \ldots$ (integer valued),

(ii) If $s < t$, then $N(s) \leq N(t)$.

(iii) For $s < t$, $N(t) - N(s) =$ number of events that occur in the interval $(s, t]$.

Definition.

A process $\{X(t), t \geq 0\}$ is said to have *stationary increments* if for any $t > s$, the distribution of $X(t) - X(s)$ depends on s and t only through the difference $t - s$, for all $s < t$.
That is, $X(t + a) - X(s + a)$ has the same distribution as $X(t) - X(s)$ for any constant a.

Definition.

A process $\{X(t), t \geq 0\}$ is said to have *independent increments* if for any $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k$, the random variable $X(t_1) - X(s_1)$, $X(t_2) - X(s_2), \ldots, X(t_k) - X(s_k)$ are independent, i.e. the numbers of events that occur in **disjoint** time intervals are **independent**.

Example. Modified simple random walk $\{X_n, n \geq 0\}$ is a process with independent and stationary increment, since $X_n = \sum_{k=0}^{n} \xi_k$ where ξ_k's are i.i.d with $\mathrm{P}(\xi_k = 1) = p$ and $\mathrm{P}(\xi_k = 0) = 1 - p$.

Definition 5.1 of Poisson Processes

A Poisson process with rate $\lambda > 0$ $\{N(t), t \geq 0\}$ is a counting process satisfying

(i) $N(0) = 0$,

(ii) For $s < t$, $N(t) - N(s)$ is independent of $N(s)$ (independent increment)

(iii) For $s < t$, $N(t) - N(s) \sim Poi(\lambda(t-s))$, i.e.,

$$\mathrm{P}(N(t) - N(s) = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$$

Remark: In (iii), the distribution of $N(t) - N(s)$ depends on $t - s$ only, not s, which implies $N(t)$ has stationary increment.

Definition 5.3 of Poisson Processes

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) The process has stationary and independent increments.

(iii) $\mathrm{P}(N(h) = 1) = \lambda h + o(h)$.

(iv) $\mathrm{P}(N(h) \geq 2) = o(h)$.

Theorem 5.1 Definitions 5.1 and 5.3 are equivalent.
[Proof of Definitions 5.1 ⇒ Definition 5.3]
From Definitions 5.1, $N(h) \sim Poi(h)$. Thus

$$\mathrm{P}(N(h) = 1) = \lambda h e^{-\lambda h} = \lambda h + o(h)$$

$$\begin{aligned} \mathrm{P}(N(h) \geq 2) &= 1 - \mathrm{P}(N(h) = 0) - \mathrm{P}(N(h) = 1) \\ &= 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} = o(h) \end{aligned}$$

Proof of Definitions 5.3 ⇒ Definition 5.1:
See textbook.

Arrival & Interarrival Times of Poisson Processes

Let

$$\begin{aligned}
S_n &= \text{Arrival time of the } n\text{-th event}, n = 1, 2, \ldots \\
T_1 &= S_1 = \text{Time until the 1st event occurs} \\
T_n &= S_n - S_{n-1} \\
&= \text{time elapsed between the } (n-1)\text{st and } n\text{-th event}, \\
&\quad n = 2, 3, \ldots
\end{aligned}$$

Proposition 5.1

The interarrival times $T_1, T_2, \ldots, T_k, \ldots,$ are i.i.d $\sim Exp(\lambda)$.

Consequently, as the distribution of the sum of n i.i.d Exp(λ) is $Gamma(n, \lambda)$, the arrival time of the nth event is

$$S_n = \sum_{i=1}^{n} T_i \sim Gamma(n, \lambda)$$

Proof of Proposition 5.1

$$
\begin{aligned}
&\mathrm{P}(T_{n+1} > t | T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
&= \mathrm{P}(0 \text{ event in } (s_n, s_n + t] | T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{where } s_n = t_1 + t_2 + \cdots + t_n) \\
&= \mathrm{P}(0 \text{ event in } (s_n, s_n + t]) \quad (\text{by indep increment}) \\
&= \mathrm{P}(N(s_n + t) - N(s_n) = 0) \\
&= e^{-\lambda t}
\end{aligned}
$$

where the last step comes from the fact that

- $N(s_n + t) - N(s_n) \sim$ Poisson(λt) and
- $P(N = k) = e^{-\mu}\mu^k/k!$ if $N \sim$ Poisson(μ), $k = 0, 1, 2 \ldots$

This shows that T_{n+1} is $\sim Exp(\lambda)$, and is independent of $T_1, T_2, \ldots, T_n$.

Definition 3 of the Poisson Process

A continuous-time stochastic process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

(i) $N(0) = 0$,

(ii) $N(t)$ counts the number of events that have occurred up to time t (i.e., it is a counting process).

(iii) The times between events are independent and identically distributed with an Exp(λ) distribution.

We have seen how Definition 5.1 implies (i), (ii) and (iii) in Definition 3. The proof of the converse is omitted.

Properties of Poisson Processes

Outline:

- Conditional Distribution of the Arrival Times
- Superposition & Thinning
- "Converse" of Superposition & Thinning

Conditional Distribution of Arrival Times is Uniform

Given $N(t) = 1$, then T_1, the arrival time of the first event $\sim$ Uniform$(0, t)$
Proof. For $s < t$,

$$\begin{aligned}
\mathrm{P}(T_1 \le s | N(t){=}1) &= \frac{\mathrm{P}(T_1 \le s, N(t) = 1)}{\mathrm{P}(N(t) = 1)} \\
&= \frac{\mathrm{P}(\text{1 event in } (0, s], \text{ no events in } (s, t])}{\mathrm{P}(N(t) = 1)} \\
&= \frac{\mathrm{P}(N(s){=}1)\mathrm{P}(N(t){-}N(s){=}0)}{\mathrm{P}(N(t) = 1)} \text{ by indep. increment} \\
&=^* \frac{(\lambda s e^{-\lambda s})(e^{-\lambda(t-s)})}{\lambda t e^{-\lambda t}} = \frac{s}{t}, \quad s < t.
\end{aligned}$$

where the step $=^*$ comes from the fact that

- $N(s) \sim$ Poisson(λs), $N(t) - N(s) \sim$ Poisson$(\lambda(t - s))$, and $N(t) \sim$ Poisson(λt)
- $P(N = k) = e^{-\mu}\mu^k/k!$ if $N \sim$ Poisson(μ), $k = 0, 1, 2 \ldots$

Review of Order Statistics

Suppose X_1, X_2,…, X_n are i.i.d. random variables with a common density $f(x)$. Their joint density would be the product of the marginal density

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n).$$

Let $X_{(i)}$ be the ith smallest number among $X_1, X_2, \ldots, X_n$. $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ is called the order statistics of $X_1, X_2, \ldots, X_n$

- $X_{(1)}$ is the minimum
- $X_{(n)}$ is the maximum
- $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$

The joint density of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is

$$h(x_1, x_2, \ldots, x_n) = \begin{cases} n!f(x_1)f(x_2)\ldots f(x_n), & \text{if } x_1 \leq x_2 \leq \ldots \leq x_n. \\ 0 & \text{otherwise} \end{cases}$$

Example

If $U_1, U_2, \ldots, U_n$ are indep. Uniform$(0, t)$, their common density is

$$f(u) = \begin{cases} 1/t, & \text{for } 0 < u < t. \\ 0 & \text{otherwise} \end{cases}$$

The joint density of their order statistics $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ is

$$h(u_1, u_2, \ldots, u_n) = n! f(u_1) f(u_2) \ldots f(u_n) = n!(1/t)^n$$

for $0 \leq u_1 \leq u_2 \leq \ldots \leq u_n < t$ and 0 elsewhere.

Theorem 5.2

Given $N(t) = n$,

$$(S_1, S_2, \ldots, S_n) \sim (U_{(1)}, U_{(2)}, \ldots, U_{(n)})$$

where $(U_{(1)}, \ldots, U_{(n)})$ are the order statistics of $(U_1, \ldots, U_n) \sim$ i.i.d Uniform $(0, t)$, i.e., the joint conditional density of $S_1, S_2, \ldots$, S_n is

$$f(s_1, s_2, \ldots, s_n | N(t) = n) = n!/t^n,\ 0 < s_1 < s_2 < \ldots < s_n$$

Proof. The event that $S_1 = s_1$, $S_2 = s_2, \ldots$, $S_n = s_n$, $N(t) = n$ is equivalent to the event $T_1 = s_1$, $T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$. Hence, by Proposition 5.1, we have the conditional joint density of $S_1, \ldots, S_n$ given $N(t) = n$ as follows:

$$\begin{aligned} f(s_1, \ldots, s_n | N(t) = n) &= \frac{f(s_1, \ldots, s_n, N(t) = n)}{\mathrm{P}(N(t) = n)} \\ &= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \ldots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{e^{-\lambda t}(\lambda t)^n / n!} \\ &= n! t^{-n}, \quad 0 < s_1 < \ldots < s_n < t \end{aligned}$$

Example 5.21. Insurance claims comes according to a Poisson process $\{N(t)\}$ with rate λ. Let

- $S_i =$ the time of the ith claims
- $C_i =$ amount of the ith claims, i.i.d with mean μ, indep. of $\{N(t)\}$

Then the total discounted cost by time t at discount rate α is given by

$$D(t) = \sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}.$$

Then

$$\begin{aligned}
\mathbb{E}[D(t)|N(t)] &= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}\middle| N(t)\right] \overset{(5.2)}{=} \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_{(i)}}\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_i}\right] = \sum_{i=1}^{N(t)} \mathbb{E}[C_i]\mathbb{E}\left[e^{-\alpha U_i}\right] \\
&= N(t)\mu \int_0^t \frac{1}{t} e^{-\alpha x} dx = N(t)\frac{\mu}{\alpha t}(1 - e^{-\alpha t})
\end{aligned}$$

Thus $\mathbb{E}[D(t)] = \mathbb{E}[N(t)]\frac{\mu}{\alpha t}(1 - e^{\alpha t}) = \frac{\lambda\mu}{\alpha}(1 - e^{-\alpha t})$

Superposition

The sum of two independent Poisson processes with respective rates λ_1 and λ_2, called the **superposition** of the processes, is again a Poisson process but with rate $\lambda_1 + \lambda_2$.

The proof is straight forward from Definition 5.3 and hence omitted.

Remark: By repeated application of the above arguments we can see that the superposition of k independent Poisson processes with rates $\lambda_1, \cdots, \lambda_k$ is again a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.

Thinning

Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ.
At each arrival of events, it is classified as a

$$\begin{cases} \text{Type 1 event with probability} & p \qquad \text{or} \\ \text{Type 2 event with probability} & 1-p, \end{cases}$$

independently of all other events. Let

$$N_i(t) = \# \text{ of type } i \text{ events occurred during } [0,t],\ i = 1,2.$$

Note that $N(t) = N_1(t) + N_2(t)$.

Proposition 5.2

$\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$.

Furthermore, the two processes are independent.

Proof of Proposition 5.2

First observe that given $N(t) = n + m$,

$$N_1(t) \sim Binomial(n + m, p). \qquad \text{(why?)}$$

Thus $\mathrm{P}(N_1(t) = n, N_2(t) = m)$

$$\begin{aligned}
&= \mathrm{P}(N_1(t) = n, N_2(t) = m | N(t) = n + m)\mathrm{P}(N(t) = n + m) \\
&= \binom{n+m}{n} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\
&= e^{-\lambda t p} \frac{(\lambda p t)^n}{n!} e^{-\lambda t(1-p)} \frac{(\lambda(1-p)t)^m}{m!} \\
&= \mathrm{P}(N_1(t) = n)\mathrm{P}(N_2(t) = m).
\end{aligned}$$

This proves the independence of $N_1(t)$ and $N_2(t)$ and that

$$N_1(t) \sim Poisson(\lambda p t), \quad N_2(t) \sim Poisson(\lambda(1-p)t).$$

Both $\{N_1(t)\}$ and $\{N_2(t)\}$ inherit the stationary and independent increment properties from $\{N(t)\}$, and hence are both Poisson processes.

Some "Converse" of Thinning & Superposition

Consider two indep. Poisson processes $\{N_A(t)\}$ and $\{N_B(t)\}$ w/ respective rates λ_A and λ_B. Let

$$S_n^A = \text{arrival time of the } n\text{th } A \text{ event}$$
$$S_m^B = \text{arrival time of the } m\text{th } B \text{ event}$$

Find $\mathrm{P}(S_n^A < S_m^B)$.

Approach 1:
Observer that $S_n^A \sim Gamma(n, \lambda_A)$, $S_m^B \sim Gamma(m, \lambda_B)$ and they are independent. Thus

$$\mathrm{P}(S_n^A < S_m^B) = \int_{x<y} \lambda_A e^{-\lambda_A x} \frac{(\lambda_A x)^{n-1}}{(n-1)!} \lambda_B e^{-\lambda_B y} \frac{(\lambda_B y)^{m-1}}{(m-1)!} dx dy$$

Some "Converse" of Thinning & Superposition (Cont'd)

Let $N(t) = N_A(t) + N_B(t)$ be the superposition of the two processes. Let

$$I_i = \begin{cases} 1 & \text{if the } i\text{th event in the superpositon process is an } A \text{ event} \\ 0 & \text{otherwise} \end{cases}.$$

The I_i, $i = 1, 2, \ldots$ are i.i.d. Bernoulli(p), where $p = \dfrac{\lambda_A}{\lambda_A + \lambda_B}$.

Approach 2:

$$\mathrm{P}(S_n^A < S_1^B) = \mathrm{P}(\text{the first } n \text{ events are all } A \text{ events}) = \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^n$$

$$\begin{aligned} \mathrm{P}(S_n^A < S_m^B) &= \mathrm{P}(\text{at least } n \ A \text{ events occur before } m \ B \text{ events}) \\ &= \mathrm{P}(\text{at least } n \text{ heads before } m \text{ tails}) \\ &= \mathrm{P}(\text{at least } n \text{ heads in the first } n+m-1 \text{ tosses}) \\ &= \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^k \left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right)^{n+m-1-k} \end{aligned}$$

Proposition 5.3 (Generalization of Proposition 5.2)

Consider a Poisson process with rate λ. If an event occurs at time t will be classified as a type i event with probability $p_i(t)$, $i = 1, \ldots, k$, $\sum_i p_i(t) = 1$, for all t, independently of all other events. then

$$N_i(t) = \text{number of type } i \text{ events occurring in } [0, t], \; i = 1, \ldots, k.$$

Note $N(t) = \sum_{i=1}^{k} N_i(t)$. Then $N_i(t)$, $i = 1, \ldots, k$ are independent Poisson random variables with means $\lambda \int_0^t p_i(s) ps$.

Remark: Note $\{N_i(t), t \geq 0\}$ are NOT Poisson processes.

Example

- Policyholders of a certain insurance company have accidents occurring according to a Poisson process with rate λ.
- The amount of time T from when the accident occurs until a claim is made has distribution $G(t) = \mathrm{P}(T \le t)$.
- Let $N_c(t)$ be the number of claims made by time t.

Find the distribution of $N_c(t)$.

Solution. Suppose an accident occurred at time s. It is claimed by time t if $s + T \le t$, i.e., with probability

$$p(s) = \mathrm{P}(T \le t - s) = G(t - s).$$

We call an accident type I if it's completed before t, and type II otherwise. By Proposition 5.3, $N_c(t)$ has a Poisson distribution with mean

$$\lambda \int_0^t p(s) ps = \lambda \int_0^t G(t - s) ds = \lambda \int_0^t G(s) ds$$

5.4.1 Nonhomogeneous Poisson Process

Definition 5.4a. A nonhomogeneous (a.k.a. non-stationary) Poisson process with intensity function $\lambda(t) \geq 0$ is a counting process $\{N(t), t \geq 0\}$ satisfying

(i) $N(0) = 0$.
(ii) having independent increments.
(iii) $\mathrm{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$.
(iv) $\mathrm{P}(N(t+h) - N(t) \geq 2) = o(h)$.

Definition 5.4b. A nonhomogeneous Poisson process with intensity function $\lambda(t) \geq 0$ is a counting process $\{N(t), t \geq 0\}$ satisfying

(i) $N(0) = 0$,
(ii) for $s, t \geq 0$, $N(t+s) - N(s)$ is independent of $N(s)$ (independent increment)
(iii) For $s, t \geq 0$, $N(t+s) - N(s) \sim Poisson(m(t+s) - m(s))$, where $m(t) = \int_0^t \lambda(u)du$

The two definitions are equivalent.

The Interarrival Times of a Nonhomogeneous Poisson Process Are NOT Independent

A nonhomogeneous Poisson process **has independent increment** but its **interarrival times** between events are

- neither independent
- nor identically distributed.

Proof. Homework.

Proposition 5.4

Let $\{N_1(t), t \geq 0\}$, and $\{N_2(t), t \geq 0\}$ be two independent nonhomogeneous Poisson process with respective intensity functions $\lambda_1(t)$ and $\lambda_2(t)$, and let $N(t) = N_1(t) + N_2(t)$. Then

(a) $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda_1(t) + \lambda_2(t)$.

(b) Given that an event of the $\{N(t), t \geq 0\}$ process occurs at time t then, independent of what occurred prior to t , the event at t was from the $\{N_1(t)\}$ process with probability

$$\frac{\lambda_1(t)}{\lambda_1(t) + \lambda_2(t)}.$$

5.4.2 Compound Poisson Processes

Definition. Let $\{N(t)\}$ be a (homogeneous) Poisson process with rate λ and Y_1, $Y_2, \ldots$ are i.i.d random variables independent of $\{N(t)\}$. The process

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

is called a *compound Poisson process*, in which $X(t)$ is defined as 0 if $N(t) = 0$.

A compound Poisson process has

- **independent increment**, since $X(t+s) - X(s) = \sum_{i=1}^{N(t+s)-N(s)} Y_{i+N(s)}$ is independent of $X(s) = \sum_{i=1}^{N(s)} Y_i$, and
- **stationary increment**, since $X(t+s) - X(s) = \sum_{i=1}^{N(t+s)-N(s)} Y_{i+N(s)}$ has the same distribution as $X(t) = \sum_{i=1}^{N(t)} Y_i$

The Mean of a Compound Poisson Process

Suppose $\mathbb{E}[Y_i] = \mu_Y$, $\mathrm{V}ar(Y_i) = \sigma_Y^2$. Note that $\mathbb{E}[N(t)] = \lambda t$.

$$\begin{aligned}
\mathbb{E}[X(t)|N(t)] &= \sum\nolimits_{i=1}^{N(t)} \mathbb{E}[Y_i|N(t)] \\
&= \sum\nolimits_{i=1}^{N(t)} \mathbb{E}[Y_i] \quad \text{(since } Y_i\text{'s are indep. of } N(t)) \\
&= N(t)\mu_Y
\end{aligned}$$

Thus

$$\mathbb{E}[X(t)] = \mathbb{E}[\mathbb{E}[X(t)|N(t)]] = \mathbb{E}[N(t)]\mu_Y = \lambda t \mu_Y$$

Variance of a Compound Poisson Process (Cont'd)

Similarly, using that $\mathbb{E}[N(t)] = \mathrm{V}ar(N(t)) = \lambda t$, we have

$$\begin{aligned}
\mathrm{V}ar[X(t)|N(t)] &= \mathrm{V}ar\left(\sum_{i=1}^{N(t)} Y_i \Big| N(t)\right) \\
&= \sum_{i=1}^{N(t)} \mathrm{V}ar(Y_i|N(t)) \\
&= \sum_{i=1}^{N(t)} \mathrm{V}ar(Y_i) \quad \text{(since } Y_i\text{'s are indep. of } N(t)\text{)} \\
&= N(t)\sigma_Y^2 \\
\mathbb{E}[\mathrm{V}ar(X(t)|N(t))] &= \mathbb{E}[N(t)\sigma_Y^2] = \lambda t \sigma_Y^2 \\
\mathrm{V}ar(\mathbb{E}[X(t)|N(t)]) &= \mathrm{V}ar(N(t)\mu_Y) = \mathrm{V}ar(N(t))\mu_Y^2 = \lambda t \mu_Y^2
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{V}ar(X(t)) &= \mathbb{E}[\mathrm{V}ar[X(t)|N(t)]] + \mathrm{V}ar(\mathbb{E}[X(t)|N(t)]) \\
&= \lambda t(\sigma_Y^2 + \mu_Y^2) = \lambda t \mathbb{E}[Y_i^2]
\end{aligned}$$

CLT of a Compound Poisson Process

As $t \to \infty$, the distribution of

$$\frac{X(t) - \mathbb{E}[X(t)]}{\sqrt{\mathrm{V}ar(X(t))}} = \frac{X(t) - \lambda t \mu_Y}{\sqrt{\lambda t(\sigma_Y^2 + \mu_Y^2)}}$$

converges to a standard normal distribution $N(0, 1)$.

5.4.3 Conditional Poisson Processes

Definition. A *conditional* (or *mixed*) *Poisson process* $\{N(t), t \geq 0\}$ is a counting process satisfying

(i) $N(0) = 0$,

(ii) having stationary increment, and

(iii) there is a random variable $\Lambda > 0$ with probability density $g(\lambda)$, such that given $\Lambda = \lambda$,

$$N(t+s) - N(s) \sim Poisson(\lambda t),$$

i.e.,

$$\mathrm{P}(N(t+s) - N(s) = k) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(\lambda) d\lambda,\ k = 0, 1, \ldots$$

Remark: In general, a conditional Poisson process does NOT have independent increment.

$$\begin{aligned}
&\mathrm{P}(N(s) = j, N(t+s) - N(s) = k) \\
&= \int_0^\infty e^{-\lambda s} \frac{(\lambda s)^j}{j!} e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(\lambda) d\lambda \\
&\neq \left(\int_0^\infty e^{-\lambda s} \frac{(\lambda s)^j}{j!} g(\lambda) d\lambda \right) \left(\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(\lambda) d\lambda \right) \\
&= \mathrm{P}(N(s) = j) \mathrm{P}(N(t+s) - N(s) = k)
\end{aligned}$$