Convex Sets

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Outline

- Definition and examples of convex sets
- Convex hulls and convex cones
- Operations that preserve convexity

Convex Set

Definition 1

A set $C \subseteq \mathbb{R}^n$ is called convex if for any $\boldsymbol{x}, \boldsymbol{y} \in C$ and $\lambda \in [0, 1]$, the point $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$ belongs to C.

- The above definition is equivalent to saying that for any $\boldsymbol{x}, \boldsymbol{y} \in C$, the line segment $[\boldsymbol{x}, \boldsymbol{y}]$ is also in C.

Examples of Convex Sets

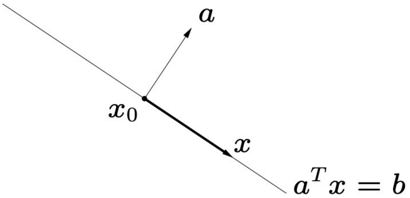
- $\emptyset, \mathbb{R}^n$.
- Lines: A line in $\mathbb{R}^n$ is a set of the form

$$L = \{\boldsymbol{z} + t\boldsymbol{d} : t \in \mathbb{R}\},$$

  where $\boldsymbol{z}, \boldsymbol{d} \in \mathbb{R}^n$ and $\boldsymbol{d} \neq \boldsymbol{0}$.

Hyperplane

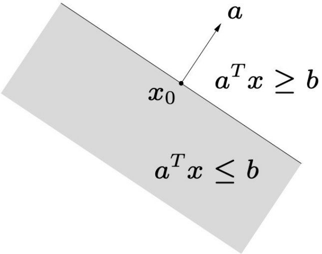
set of the form $\{x \mid a^\top x = b\}$ ($a \neq 0$)



- a is the normal vector
- hyperplanes are affine and convex

Halfspace

set of the form $\{x \mid a^\top x \le b\}$ ($a \ne 0$)



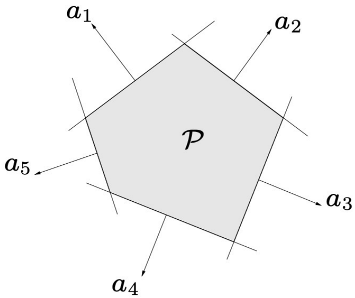
- a is the normal vector
- halfspaces are convex

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Convexity of Balls

Lemma 2

Let $\boldsymbol{c} \in \mathbb{R}^n$ *and* $r > 0$. *Then the open ball*

$$B(\boldsymbol{c}, r) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{c}\| < r\}$$

and the closed ball

$$B[\boldsymbol{c}, r] = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{c}\| \leq r\}$$

are convex.

Note that the norm is an arbitrary norm defined over $\mathbb{R}^n$.

Norm balls

norm: a function $\|\cdot\|_{\text{norm}}$ that satisfies

- $\|x\|_{\text{norm}} \geq 0$; $\|x\|_{\text{norm}} = 0$ if and only if $x = 0$
- $\|tx\|_{\text{norm}} = |t| \, \|x\|_{\text{norm}}$ for $t \in \mathbb{R}$
- $\|x + y\|_{\text{norm}} \leq \|x\|_{\text{norm}} + \|y\|_{\text{norm}}$

notation: $\|\cdot\|_{\text{norm}}$ is general (unspecified) norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\|_{\text{norm}} \leq r\}$

Convexity of Ellipsoids

An ellipsoid is a set of the form

$$E = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + 2\boldsymbol{b}^T \boldsymbol{x} + c \leq 0\},$$

where $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ is positive semidefinite, $\boldsymbol{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Lemma 3

E is convex.

Proof

- Write E as $E = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \leq 0\}$ where $f(\boldsymbol{x}) \equiv \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + 2\boldsymbol{b}^T \boldsymbol{x} + c$.
- Take $\boldsymbol{x}, \boldsymbol{y} \in E$ and $\lambda \in [0, 1]$. Then $f(\boldsymbol{x}) \leq 0, f(\boldsymbol{y}) \leq 0$.
- The vector $\boldsymbol{z} = \lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$ satisfies $\boldsymbol{z}^T \boldsymbol{Q} \boldsymbol{z} = \lambda^2 \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + (1 - \lambda)^2 \boldsymbol{y}^T \boldsymbol{Q} \boldsymbol{y} + 2\lambda(1 - \lambda)\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{y}$.
- $\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{y} \leq \|\boldsymbol{Q}^{1/2} \boldsymbol{x}\| \cdot \|\boldsymbol{Q}^{1/2} \boldsymbol{y}\| = \sqrt{\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}} \sqrt{\boldsymbol{y}^T \boldsymbol{Q} \boldsymbol{y}} \leq \frac{1}{2}(\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{y}^T \boldsymbol{Q} \boldsymbol{y})$
- $\boldsymbol{z}^T \boldsymbol{Q} \boldsymbol{z} \leq \lambda \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}^T \boldsymbol{Q} \boldsymbol{y}$

$$
\begin{aligned}
f(\boldsymbol{z}) &= \boldsymbol{z}^T \boldsymbol{Q} \boldsymbol{z} + 2\boldsymbol{b}^T \boldsymbol{z} + c \\
&\leq \lambda \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}^T \boldsymbol{Q} \boldsymbol{y} + 2\lambda \boldsymbol{b}^T \boldsymbol{x} + 2(1 - \lambda)\boldsymbol{b}^T \boldsymbol{y} + \lambda c + (1 - \lambda)c \\
&= \lambda f(\boldsymbol{x}) + (1 - \lambda) f(\boldsymbol{y}) \leq 0,
\end{aligned}
$$

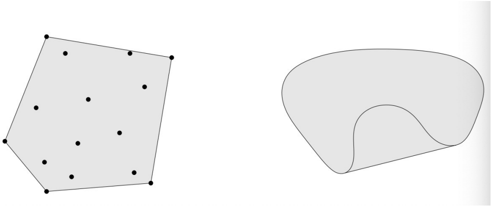
Convex combination and convex hull

convex combination of x_1, $\cdots$, x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull conv(S): set of all convex combinations of points in S



The Probability Simplex: Definition

The **unit simplex** (or probability simplex) in $\mathbb{R}^n$ is the set of all probability distributions over n discrete outcomes.

Definition 4

The probability simplex Δ_{n-1} is defined as:

$$\Delta_{n-1} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0 \text{ for } i = 1, \ldots, n \right\}.$$

- **Dimension:** Although it lives in $\mathbb{R}^n$, it is an $(n-1)$-dimensional object because of the equality constraint $\mathbf{1}^T \boldsymbol{x} = 1$.

Geometry of the Simplex

Visualizing the simplex helps students understand how constraints "shape" space:

- $n = 2$ **(Δ_1):** A line segment in $\mathbb{R}^2$ connecting $(1, 0)$ and $(0, 1)$.
- $n = 3$ **(Δ_2):** An equilateral triangle in $\mathbb{R}^3$ with vertices at $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$.

Key Observation

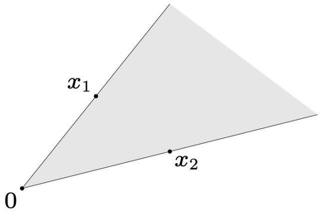
The simplex is the **convex hull** of the standard basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$.

Convex cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0, \theta_2 \geq 0$



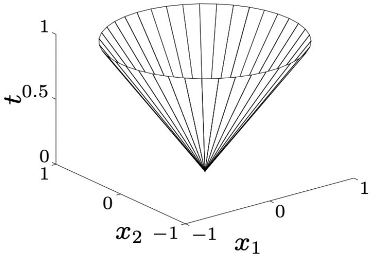
convex cone: set that contains all conic combinations of points in the set

Norm cones

norm cone: $\{(x,t) \mid \|x\|_{\text{norm}} \le t\}$

Euclidean-norm cone $\{(x,t) \mid \|x\| \le t\}$ is called **second-order cone**

norm balls and norm cones are convex



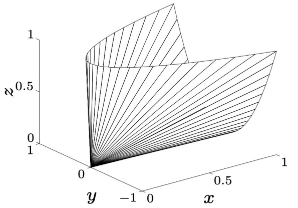
Positive semidefinite cone

- $\mathbb{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \succeq 0$: positive semidefinite $n \times n$ matrices

$$X \in \mathbb{S}^n_+ \qquad \Longleftrightarrow \qquad z^\top X z \geq 0 \text{ for all } z$$

- $\mathbb{S}^n_+$ is a convex cone
- $\mathbb{S}^n_{++} = \{X \in \mathbb{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices }

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2_+$



boundary of $\mathbb{S}^2_+$

The Normal Cone: Definition

The normal cone describes the set of all vectors that are "outward-pointing" and perpendicular to a supporting hyperplane at a specific point.

Definition 5

Let $C \subseteq \mathbb{R}^n$ be a set. The **normal cone** to C at a point $\boldsymbol{x} \in C$ is defined as:

$$N_C(\boldsymbol{x}) = \{\boldsymbol{g} \in \mathbb{R}^n \mid \boldsymbol{g}^T(\boldsymbol{y} - \boldsymbol{x}) \leq 0, \quad \forall \boldsymbol{y} \in C\}$$

- **Intuition:** For any vector $\boldsymbol{y} \in C$, the vector $(\boldsymbol{y} - \boldsymbol{x})$ points "into" the set. A normal vector $\boldsymbol{g}$ must make an **obtuse angle** $(\geq 90^\circ)$ with every such inward-pointing vector.
- If $\boldsymbol{x}$ is in the **interior** of C, then $N_C(\boldsymbol{x}) = \{\mathbf{0}\}$.

Geometry of the Normal Cone

The shape of the normal cone depends on the "sharpness" of the boundary at x:

- **Smooth Boundary:** If the boundary is smooth at x, the normal cone is simply a **ray** (a line from the origin) pointing in the direction of the outward normal.
- **Corners/Vertices:** If x is a corner (like in a polyhedron), the normal cone is "fat"—it contains all vectors that lie between the normals of the adjacent faces.

Key Property

$N_C(x)$ is always a **closed convex cone**, regardless of whether C is smooth or not.

Intersection

intersection of (any number of) convex sets is convex

example: positive semidefinite cone $\mathbb{S}^n_+$

note that $\mathbb{S}^n_+$ can be expressed as

$$\mathbb{S}^n_+ = \bigcap_{z \neq 0} \{X \in \mathbb{S}^n \mid z^\top X z \geq 0\}$$

for each $z \neq 0$, $z^\top X z$ is linear function of X, so

$$S_z := \{X \in \mathbb{S}^n \mid z^\top X z \geq 0\} \quad \text{is convex}$$

therefore, $\mathbb{S}^n_+ = \bigcap_{z \neq 0} S_z$ is also convex
Another example: polyhedra

Affine function

suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine (i.e., $f(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(X) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex}$$

Solution Set to LMI is Convex

A **Linear Matrix Inequality (LMI)** in $\boldsymbol{x} \in \mathbb{R}^n$ is an expression of the form:

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{x}_1 \boldsymbol{A}_1 + \boldsymbol{x}_2 \boldsymbol{A}_2 + \cdots + \boldsymbol{x}_n \boldsymbol{A}_n \preceq \boldsymbol{B}$$

where $\boldsymbol{B}, \boldsymbol{A}_i \in \mathbb{S}^m$ are symmetric matrices, and $\preceq$ denotes matrix inequality ($M \preceq N \iff N - M \in \mathbb{S}^m_+$).

Lemma 6

The solution set $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}(\boldsymbol{x}) \preceq \boldsymbol{B}\}$ is convex.

Proof using Affine Mapping:

1. Define the function $f : \mathbb{R}^n \to \mathbb{S}^m$ as $f(\boldsymbol{x}) = \boldsymbol{B} - \sum_{i=1}^n \boldsymbol{x}_i \boldsymbol{A}_i$.
2. Note that f is an **affine function** of $\boldsymbol{x}$.
3. The set S can be rewritten as the **inverse image** of the PSD cone:

$$S = \{\boldsymbol{x} \mid f(\boldsymbol{x}) \in \mathbb{S}^m_+\} = f^{-1}(\mathbb{S}^m_+)$$

4. Since the PSD cone $\mathbb{S}^m_+$ is convex and the inverse image of a convex set under an affine mapping is convex, S is convex.