Interior Point Method

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Outline

- Newton's method: unconstrained and equality-constrained optimization
- Interior-point idea for inequality constraints
- Central path and duality-gap interpretation
- Barrier algorithm and parameter update
- Feasible initialization strategies

Review: Newton in the Unconstrained Case

$$\min_{\boldsymbol{x}} f(\boldsymbol{x}).$$

At iterate $\boldsymbol{x}$, use local quadratic model:

$$f(\boldsymbol{x} + \Delta\boldsymbol{x}) \approx f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top \Delta\boldsymbol{x} + \frac{1}{2}\Delta\boldsymbol{x}^\top \nabla^2 f(\boldsymbol{x}) \Delta\boldsymbol{x}.$$

Choose $\Delta\boldsymbol{x}$ to minimize this quadratic:

$$\Delta\boldsymbol{x}_{\mathrm{nt}} = -\nabla^2 f(\boldsymbol{x})^{-1} \nabla f(\boldsymbol{x}).$$

Then update $\boldsymbol{x}^+ = \boldsymbol{x} + \alpha\Delta\boldsymbol{x}_{\mathrm{nt}}$.

Part I: Equality-Constrained Setup

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{aligned}$$

with f convex and twice differentiable.

Assumptions (typical):

- $\boldsymbol{A} \in \mathbb{R}^{p \times n}$ has full row rank,
- feasible set $\mathcal{C} = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ is nonempty,
- $\nabla^2 f(\boldsymbol{x})$ is positive definite on $\mathcal{N}(\boldsymbol{A})$.

This problem is the inner engine of barrier methods.

Extend to Equality Constraints

Newton direction solves

$$\min_{\Delta \boldsymbol{x}} \left(\nabla f(\boldsymbol{x})^\top \Delta \boldsymbol{x} + \frac{1}{2} \Delta \boldsymbol{x}^\top \nabla^2 f(\boldsymbol{x}) \Delta \boldsymbol{x} \right) \quad \text{s.t. } \boldsymbol{A} \Delta \boldsymbol{x} = 0.$$

Its KKT conditions are

$$\begin{aligned} \nabla^2 f(\boldsymbol{x}) \Delta \boldsymbol{x} + \nabla f(\boldsymbol{x}) + \boldsymbol{A}^\top \boldsymbol{w} &= 0, \\ \boldsymbol{A} \Delta \boldsymbol{x} &= 0. \end{aligned}$$

$$\iff \begin{bmatrix} \nabla^2 f(\boldsymbol{x}) & \boldsymbol{A}^\top \\ \boldsymbol{A} & 0 \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{x} \\ \boldsymbol{w} \end{bmatrix} = - \begin{bmatrix} \nabla f(\boldsymbol{x}) \\ \boldsymbol{0} \end{bmatrix}.$$

Algorithm: Equality-Constrained Newton (Feasible-Start)

Input: feasible $\boldsymbol{x}^{(0)}$ with $\boldsymbol{A}\boldsymbol{x}^{(0)} = \boldsymbol{b}$, tolerance ϵ.

For $k = 0, 1, 2, \ldots$:

1. Solve for $(\Delta\boldsymbol{x}^{(k)}, \boldsymbol{w}^{(k)})$:

$$\begin{bmatrix} \nabla^2 f(\boldsymbol{x}^{(k)}) & \boldsymbol{A}^\top \\ \boldsymbol{A} & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x}^{(k)} \\ \boldsymbol{w}^{(k)} \end{bmatrix} = - \begin{bmatrix} \nabla f(\boldsymbol{x}^{(k)}) \\ \boldsymbol{0} \end{bmatrix}.$$

2. Choose $\alpha_k \in (0, 1]$ by backtracking line search.
3. Update $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k \Delta\boldsymbol{x}^{(k)}$.
4. Stop when Newton decrement is below ϵ.

Fact 1 (Feasibility Preservation)

Suppose $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *and the Newton direction satisfies* $\boldsymbol{A}\Delta\boldsymbol{x} = 0$*. Then for any step size* α*,*

$$\boldsymbol{A}(\boldsymbol{x} + \alpha\Delta\boldsymbol{x}) = \boldsymbol{b}.$$

- This gives a simple residual structure for feasible-start Newton.
- The main limitation is the need for an initial feasible point.

Inequality-Constrained Problem Setup

Now consider

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f_0(\boldsymbol{x}) \\ \text{s.t.} \quad & f_i(\boldsymbol{x}) \le 0,\ i = 1, \ldots, m, \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned}$$

- Hard part is handling the boundary $f_i(\boldsymbol{x}) = 0$ smoothly.
- Interior-point idea: stay strictly inside $f_i(\boldsymbol{x}) < 0$, then approach the boundary only near the end.

Log Barrier as Indicator Approximation

For one scalar inequality $u < 0$, define the indicator

$$I_{-}(u) = \begin{cases} 0, & u < 0, \\ +\infty, & u \geq 0. \end{cases}$$

Approximate $I_{-}(u)$ by

$$\frac{1}{t}(-\log(-u)), \qquad t > 0.$$

As t grows, this approximation gets tighter near the feasible region.

For m inequalities, define

$$\phi(\boldsymbol{x}) = -\sum_{i=1}^{m} \log(-f_i(\boldsymbol{x})).$$

Barrier Subproblem

For each $t > 0$, solve

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & t f_0(\boldsymbol{x}) + \phi(\boldsymbol{x}) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned}$$

$$\mathrm{dom}(\phi) = \{\boldsymbol{x} \mid f_i(\boldsymbol{x}) < 0,\ i = 1, \ldots, m\}.$$

- ϕ keeps iterates strictly interior.
- Larger t: more weight on minimizing f_0, less on centrality.

For $\phi(\boldsymbol{x}) = -\sum_i \log(-f_i(\boldsymbol{x}))$:

$$\nabla \phi(\boldsymbol{x}) = \sum_{i=1}^{m} \frac{1}{-f_i(\boldsymbol{x})} \nabla f_i(\boldsymbol{x}),$$

$$\nabla^2 \phi(\boldsymbol{x}) = \sum_{i=1}^{m} \frac{1}{f_i(\boldsymbol{x})^2} \nabla f_i(\boldsymbol{x}) \nabla f_i(\boldsymbol{x})^\top \; + \; \sum_{i=1}^{m} \frac{1}{-f_i(\boldsymbol{x})} \nabla^2 f_i(\boldsymbol{x}).$$

- If f_i are affine, second summation vanishes.
- Hessian blows up near boundary, creating repulsion from $f_i(\boldsymbol{x}) = 0$.

Central Path

Let $\boldsymbol{x}^\star(t)$ solve

$$\min_{\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}} \; t f_0(\boldsymbol{x}) + \phi(\boldsymbol{x}).$$

Definition 2 (Central Path)

$$\{\boldsymbol{x}^\star(t) : t > 0\}.$$

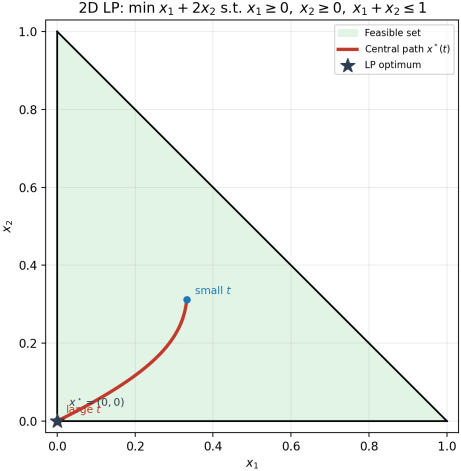
As t increases:

- objective term $t f_0$ dominates,
- barrier influence weakens,
- points move toward true constrained optimum.

2D LP Example: Visualizing the Central Path

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} \quad x_1 + 2x_2$$

$$\text{s.t.} \quad x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1.$$



Dual Variables from Barrier Stationarity

At $\boldsymbol{x}^\star(t)$, stationarity of barrier problem gives

$$t\nabla f_0(\boldsymbol{x}^\star) + \sum_{i=1}^{m} \frac{1}{-f_i(\boldsymbol{x}^\star)} \nabla f_i(\boldsymbol{x}^\star) + \boldsymbol{A}^\top \boldsymbol{\nu}^\star(t) = 0.$$

Define

$$\lambda_i^\star(t) = \frac{1}{-t f_i(\boldsymbol{x}^\star(t))} > 0.$$

Then the stationarity condition can be written as

$$\nabla f_0(\boldsymbol{x}^\star) + \sum_{i=1}^{m} \lambda_i^\star \nabla f_i(\boldsymbol{x}^\star) + \boldsymbol{A}^\top \frac{\boldsymbol{\nu}^\star(t)}{t} = 0,$$

$$\lambda_i^\star(t)(-f_i(\boldsymbol{x}^\star(t))) = \frac{1}{t}.$$

Perturbed Complementarity Interpretation

For each $t > 0$, the central-path pair satisfies

$$\lambda_i^\star(t)(-f_i(\boldsymbol{x}^\star(t))) = \frac{1}{t}.$$

- This is a smooth relaxation of exact complementarity $\lambda_i(-f_i(\boldsymbol{x})) = 0$.
- As $t \to \infty$, the right-hand side tends to 0, so the KKT conditions are recovered.

Duality Gap Bound m/t

For central-path point $\boldsymbol{x}^\star(t)$, one obtains

$$f_0(\boldsymbol{x}^\star(t)) - p^\star \leq \frac{m}{t}.$$

Key consequence:

- m/t is an explicit upper bound on suboptimality,
- gives practical stopping criterion independent of unknown $p^\star$.

Fact 3 (Stopping Rule)

If $m/t \leq \epsilon$, then $\boldsymbol{x}^\star(t)$ is ϵ-suboptimal.

Newton Step for Barrier Subproblem

Let $F_t(\boldsymbol{x}) = t f_0(\boldsymbol{x}) + \phi(\boldsymbol{x})$. Solve equality-constrained Newton system:

$$\begin{bmatrix} \nabla^2 F_t(\boldsymbol{x}) & \boldsymbol{A}^\top \\ \boldsymbol{A} & 0 \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{x} \\ \Delta \boldsymbol{\nu} \end{bmatrix} = - \begin{bmatrix} \nabla F_t(\boldsymbol{x}) + \boldsymbol{A}^\top \boldsymbol{\nu} \\ \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \end{bmatrix}.$$

Backtracking chooses α to ensure:

- strict interiority: $f_i(\boldsymbol{x} + \alpha \Delta \boldsymbol{x}) < 0$,
- sufficient decrease in residual/merit.

Barrier Method (Full Algorithm)

1. choose strictly feasible $\boldsymbol{x}$, initial $t_0 > 0$, growth factor $\mu > 1$,
2. for current t, run equality-constrained Newton on F_t until inner tolerance,
3. update $t \leftarrow \mu t$,
4. stop when $m/t \leq \epsilon$.

Approximate outer-iteration count:

$$N_{\text{outer}} \approx \frac{\log(m/(\epsilon t_0))}{\log \mu}.$$

Choosing μ and Inner Accuracy

Tradeoff:

- large μ: fewer outer iterations, harder inner solves,
- small μ: easier inner solves, more outer iterations.

Common practical strategy:

- moderately large μ (e.g., 10–20),
- solve each inner Newton system to medium precision,
- tighten tolerances near the end.

Total runtime depends mostly on linear-system solves, not iteration count alone.

Need for a Strictly Feasible Start

Classical barrier method requires

$$f_i(\boldsymbol{x}) < 0, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

at initialization.

- Equality feasibility can be enforced by projection/elimination.
- Strict inequality feasibility is the main difficulty.

This motivates Phase I procedures.

Phase I Problem for Feasibility

Solve

$$\begin{aligned} \min_{\boldsymbol{x},s} \quad & s \\ \text{s.t.} \quad & f_i(\boldsymbol{x}) \le s, \quad i = 1, \dots, m, \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned}$$

Interpretation:

- s shifts all inequality thresholds uniformly,
- minimizing s tries to push all constraints strictly below zero.

How Phase I Certifies Feasibility

Let $(\boldsymbol{x}^{\mathrm{I}}, s^{\mathrm{I}})$ be optimal for Phase I.

- If $s^{\mathrm{I}} < 0$: $\boldsymbol{x}^{\mathrm{I}}$ is strictly feasible for original problem.
- If $s^{\mathrm{I}} = 0$: boundary feasible (not strict), typically perturb and continue.
- If $s^{\mathrm{I}} > 0$: original constraints are infeasible.

Standard workflow:

1. run Phase I,
2. if feasible, use output as start for Phase II barrier method.

Infeasible-Start Barrier Newton Alternative

Instead of explicit Phase I, use infeasible-start Newton on barrier KKT residual:

$$\boldsymbol{r}_t(\boldsymbol{x}, \boldsymbol{\nu}) = \begin{bmatrix} \nabla F_t(\boldsymbol{x}) + \boldsymbol{A}^\top \boldsymbol{\nu} \\ \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \end{bmatrix}.$$

- start with $f_i(\boldsymbol{x}) < 0$ but allow $\boldsymbol{A}\boldsymbol{x} \neq \boldsymbol{b}$,
- reduce both stationarity and feasibility residuals simultaneously,
- often simpler in solver implementations.

Summary

- Start from Newton's method for equality-constrained KKT systems.
- Introduce inequalities via the log barrier.
- Central path yields dual variables and perturbed complementarity.
- Duality gap m/t provides a direct stopping criterion.
- Feasibility is handled by Phase I or infeasible-start techniques.

References for Deeper Study

- R. Tibshirani, “Barrier method” lecture slides (CMU).
- S. Boyd and L. Vandenberghe, *Convex Optimization* slides.
- S. Boyd and L. Vandenberghe, *Convex Optimization*, Chapter 11.