

Introduction to spectral methods



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Outline

- A motivating application: community detection
- A general recipe for spectral methods and more applications

A motivating application: community detection

Community detection / graph clustering

Community structures are common in many social networks



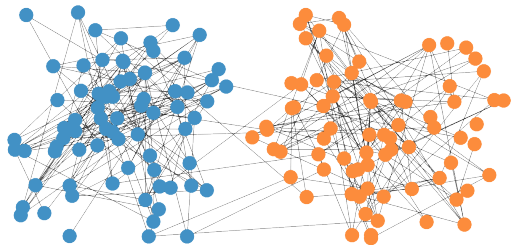
figure credit: The Future Buzz



figure credit: S. Papadopoulos

Goal: partition users into several clusters based on their friendships / similarities

A simple model: stochastic block model (SBM)

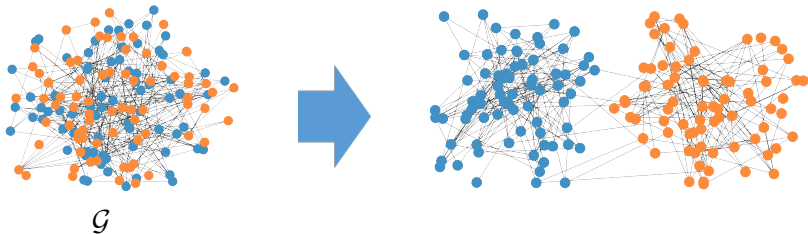


$x_i^* = 1$: 1st community

$x_i^* = -1$: 2nd community

- n nodes $\{1, \dots, n\}$
- 2 communities
- n unknown variables: $x_1^*, \dots, x_n^* \in \{1, -1\}$
 - encode community memberships

A simple model: stochastic block model (SBM)



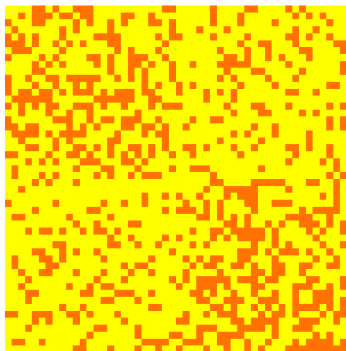
- observe a graph \mathcal{G}

$$(i, j) \in \mathcal{G} \text{ with prob. } \begin{cases} p, & \text{if } i \text{ and } j \text{ are from same community} \\ q, & \text{else} \end{cases}$$

Here, $p > q$

- **Goal:** recover community memberships of all nodes, i.e., $\{x_i^*\}$

Adjacency matrix

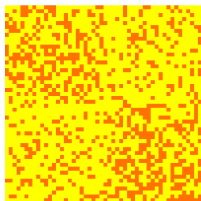


Consider the adjacency matrix $\mathbf{A} \in \{0, 1\}^{n \times n}$ of \mathcal{G} :

$$A_{i,j} = \begin{cases} 1, & \text{if } (i, j) \in \mathcal{G} \\ 0, & \text{else} \end{cases}$$

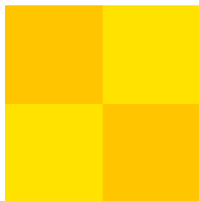
- WLOG, suppose $x_1^* = \dots = x_{n/2}^* = 1$; $x_{n/2+1}^* = \dots = x_n^* = -1$

Adjacency matrix



\mathbf{A}

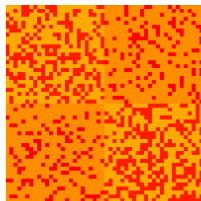
=



$\mathbb{E}[\mathbf{A}]$

rank 2

+



$\mathbf{A} - \mathbb{E}[\mathbf{A}]$

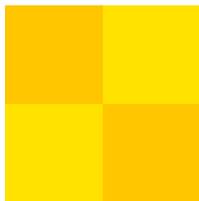
$$\mathbb{E}[\mathbf{A}] = \begin{bmatrix} p\mathbf{1}\mathbf{1}^\top & q\mathbf{1}\mathbf{1}^\top \\ q\mathbf{1}\mathbf{1}^\top & p\mathbf{1}\mathbf{1}^\top \end{bmatrix} = \underbrace{\frac{p+q}{2}\mathbf{1}\mathbf{1}^\top}_{\text{uninformative bias}} + \frac{p-q}{2} \underbrace{\begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix}}_{=\mathbf{x}^*=[x_i]_{1 \leq i \leq n}} [\mathbf{1}^\top, -\mathbf{1}^\top]$$

Spectral clustering



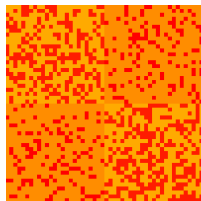
\mathbf{A}

=



$\underbrace{\mathbb{E}[\mathbf{A}]}_{\text{rank 2}}$

+



$\mathbf{A} - \mathbb{E}[\mathbf{A}]$

1. computing the leading eigenvector $\mathbf{u} = [u_i]_{1 \leq i \leq n}$ of $\mathbf{A} - \frac{p+q}{2} \mathbf{1}\mathbf{1}^\top$
2. rounding: output $x_i = \begin{cases} 1, & \text{if } u_i > 0 \\ -1, & \text{if } u_i < 0 \end{cases}$

Rationale behind spectral clustering

Recovery is reliable if $\underbrace{\mathbf{A} - \mathbb{E}[\mathbf{A}]}_{\text{perturbation}}$ is sufficiently small

- if $\mathbf{A} - \mathbb{E}[\mathbf{A}] = \mathbf{0}$, then

$$\mathbf{u} \propto \pm \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} \implies \text{perfect clustering}$$

A general recipe for spectral methods

Three key steps:

- identify a key matrix M^* , whose eigenvectors disclose crucial information
- construct a surrogate matrix M of M^* using data
- compute corresponding eigenvectors of M

Low-rank matrix completion



figure credit: Candès

- consider a low-rank matrix $M^* = U^* \Sigma^* V^{*\top}$
- each entry $M_{i,j}^*$ is observed independently with prob. p
- **intermediate goal:** estimate U^*, V^*

Spectral method for matrix completion

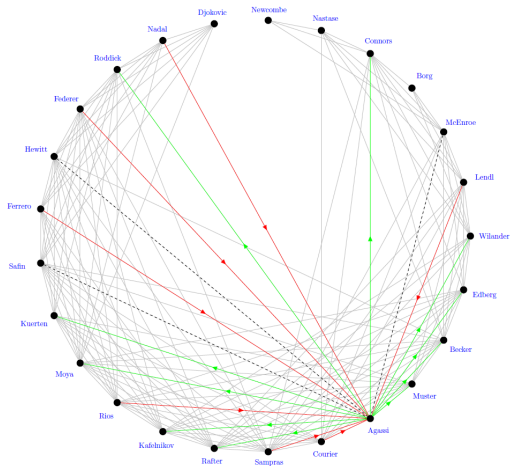
1. identify the key matrix M^*
2. construct surrogate matrix $M \in \mathbb{R}^{n \times n}$ as

$$M_{i,j} = \begin{cases} \frac{1}{p} M_{i,j}^*, & \text{if } M_{i,j}^* \text{ is observed} \\ 0, & \text{else} \end{cases}$$

- **rationale for rescaling:** ensures $\mathbb{E}[M] = M^*$

3. compute the rank- r SVD $U\Sigma V^T$ of M , and return (U, Σ, V)

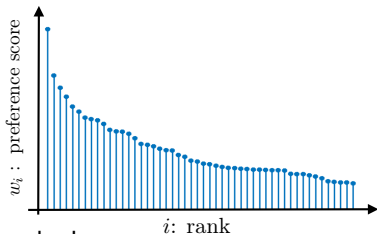
Ranking from pairwise comparisons



pairwise comparisons for ranking tennis players

figure credit: Bozóki, Csató, Temesi

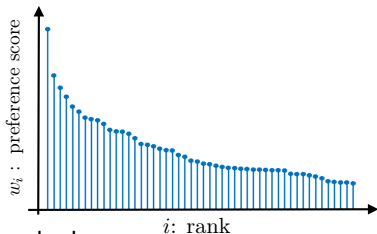
Bradley-Terry-Luce (logistic) model



- n items to be ranked
- assign a latent score $\{w_i^*\}_{1 \leq i \leq n}$ to each item, so that
item $i \succ$ item j if $w_i^* > w_j^*$
- each pair of items (i, j) is compared independently

$$\mathbb{P}\{\text{item } j \text{ beats item } i\} = \frac{w_j^*}{w_i^* + w_j^*}$$

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$$\text{item } i \succ \text{item } j \quad \text{if} \quad w_i^* > w_j^*$$

- each pair of items (i, j) is compared independently

$$y_{i,j} \stackrel{\text{ind.}}{=} \begin{cases} 1, & \text{with prob. } \frac{w_j^*}{w_i^* + w_j^*} \\ 0, & \text{else} \end{cases}$$

- **intermediate goal:** estimate score vector w^* (up to scaling)

Spectral ranking

1. identify key matrix P^* —probability transition matrix

$$P_{i,j}^* = \begin{cases} \frac{1}{n} \cdot \frac{w_j^*}{w_i^* + w_j^*}, & \text{if } i \neq j \\ 1 - \sum_{l:l \neq i} P_{i,l}^*, & \text{if } i = j \end{cases}$$

Rationale:

- P^* obeys

$$w_i^* P_{i,j}^* = w_j^* P_{j,i}^* \quad (\text{detailed balance})$$

- Thus, the stationary distribution π^* of P^* obeys

$$\pi^* = \frac{1}{\sum_l w_l^*} \mathbf{w}^* \quad (\text{reveals true scores})$$

Spectral ranking

2. construct a surrogate matrix \mathbf{P} obeying

$$P_{i,j} = \begin{cases} \frac{1}{n}y_{i,j}, & \text{if } i \neq j \\ 1 - \sum_{l:l \neq i} P_{i,l}, & \text{if } i = j \end{cases}$$

3. return leading left eigenvector $\boldsymbol{\pi}$ of \mathbf{P} as score estimate

— closely related to PageRank

Spectral ranking

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Key: stability of eigenspace against perturbation $M - M^*$?